Measurement of Time–Variant Linear Channels

G. E. Pfander and D. F. Walnut

***Abstract*— The goal of channel measurement or operator identification is to obtain complete knowledge of a channel operator by observing the image of a finite number of input signals. In this paper it is shown that the spreading support of the operator (that is, the support of the symplectic Fourier transform of the Kohn–Nirenberg symbol of the operator) has area less than one then the operator is identifiable. If the spreading support is larger than one, then the operator is not identifiable. The shape of the support region is essentially arbitrary thereby proving a conjecture of Bello. The input signal considered is a weighted delta train where the weights are the window function of a finite Gabor system whose elements satisfy a certain robust completeness property.**

***Index Terms*— Bandlimited Kohn–Nirenberg symbols, channel measurement, Gabor and time–frequency analysis, operator identification, spreading functions, underspread operators.**

I. INTRODUCTION

THE measurement of incompletely known linear channel operators based on the observation of a single input and the corresponding output signal is a traditional goal in *communications engineering*.

Starting in the late 1950s, Thomas Kailath analyzed the question whether an unknown time–varying channel operator H with a known restriction on time and frequency spread can be measured by applying the operator to a single known input signal f, that is, whether the operator H can be identified by analyzing the single channel output Hf [1], [2]. If so then we say that the class of such operators is *identifiable* with *identifier* f. Kailath considered operators formally given by

$$(Hf)(x) = \int_{-\frac{b}{2}}^{\frac{b}{2}} \int_0^a \eta_H(t,\nu)\, T_t M_\nu f(x)\, dt\, d\nu\,, \quad x \in \mathbb{R},$$

where T_t is a *time–shift* by t, that is, $T_t\, f(x) = f(x-t)$, $t \in \mathbb{R}$, and M_ν is the *frequency shift* or *modulation* given by $\widehat{M_\nu\, f}\,(\gamma) = \widehat{f}(\gamma-\nu)$, $\nu \in \widehat{\mathbb{R}} = \mathbb{R}$, that is, $M_\nu\, f(x) = e^{2\pi i \nu x}\, f(x)$, where $\widehat{f}(\gamma) = \int f(x)\, e^{-2\pi i x\gamma}\, dx$, $\gamma \in \widehat{\mathbb{R}}$.[1]The function η_H is called *spreading function* of H, a denotes the *maximal time–delay* and $\frac{b}{2}$ is the *maximal doppler spread* of H.

In the landmark paper *Time–Variant Communication Channels* [3], Kailath postulated that members in a collection of communication channels that are characterized by having common maximum delay a and common maximum Doppler spread $\frac{b}{2}$, that is, all H such that $\eta_H(t,\nu) = 0$ for $(t,\nu) \notin R = [0,a] \times [-\frac{b}{2}, \frac{b}{2}]$, would be identifiable by a single input signal if and only if the area of the rectangle R satisfies $\mathrm{vol}\,(R) = S_R = ab \leq 1$. To show the necessity of this so–called *underspread condition*, Kailath provided ingenious arguments based on the comparison of the degrees of freedom of the operator, and degrees of freedom of the output signal. To count these degrees of freedom, Kailath used the theoretical construct of a bandlimited input signal with finite duration.

Being aware of the mathematical shortcomings of his approach, and understanding the contemporary and groundbreaking work of Slepian, Landau, and Pollak on "the dimensions of the space of essentially time- and bandlimited functions" [4]–[6], Kailath conjectured that the underspread condition $ab \leq 1$ is necessary in general:

> Recent work by Landau and Shannon has shown that the concept of approximately $2TW$ degrees of freedom holds even in such cases. This leads us to believe that our proof of the necessity of the $BL \leq 1$ $[a = L, b = B]$ condition is not merely a consequence of the special properties of strictly band-limited functions. It would be valuable to find an alternative method of proof. [2]

Kailath's assertion has been proven in general only recently [7].

In the paper *Measurement of Random Time–Variant Linear Channels* [8], Philip Bello postulated that the rectangular support condition $\mathrm{vol}\,(R) = S_R = ab \leq 1$ is too restrictive. Bello considers operators given by

$$(Hf)(x) = \iint_A \eta_H(t,\nu)\, T_t M_\nu f(x)\, dt\, d\nu\,, \quad x \in \mathbb{R},$$

where A is an essentially arbitrary bounded region in the time–frequency plane $\mathbb{R}\times\widehat{\mathbb{R}}$ and postulates:

> Unfortunately the criterion $S_R < 1$ [of Kailath] has been uncritically accepted subsequently as the channel measurability criterion for random time–varying linear channels, without paying sufficiently careful attention to the conditions under which it was derived. In this paper we shall show that the criterion $S_R < 1$ is not the proper channel measurability criterion, and we shall propose a new criterion, $S_A < 1$ [S_A denotes the area of A], where the parameter S_A is called the area spread factor of the channel. [8]

In other words, Bello claimed that one could replace the rectangle spanned by maximum delay a and maximum Doppler spread $\frac{b}{2}$ by any bounded region A of time–frequency shifts. The corresponding operator class would be identifiable if the

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[1] Following standard mathematical practice, we identify the "time domain" ($\mathbb{R}$) as distinct from the "frequency domain" ($\widehat{\mathbb{R}}$) even though each symbol represents the real numbers.

area of the region is smaller than one and not identifiable if it is larger than one. Clearly, Kailath's assertions can be seen as a special case of Bello's result, namely when A is a rectangle.

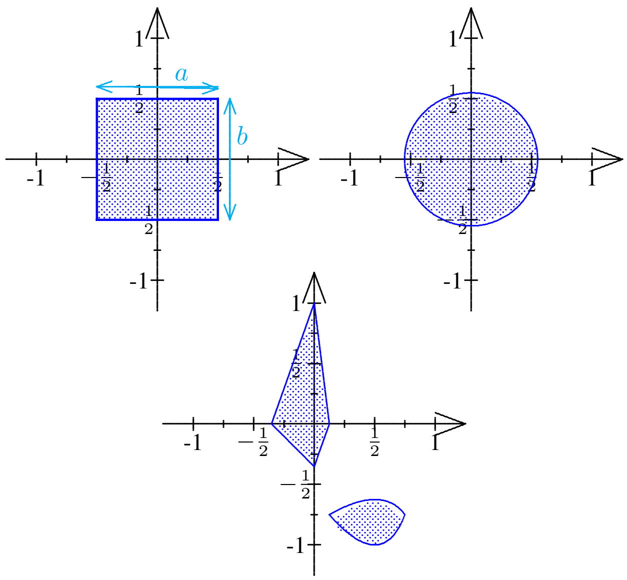

Fig. 1. Spreading support regions of area less than or equal to one which characterize identifiable operator classes.

Similar to Kailath's approach, Bello discretizes the measurement setup in order to apply dimension counting arguments. In fact, he assumes that

> the input to a channel is confined by a time gate to the time interval $0 < t < T$ and the output spectrum is confined by a bandpass filter to the frequency interval $-\frac{1}{2}W < t < \frac{1}{2}W$. [8]

Hence, Bello measures operators of the form $H_{W,T} = Q_W \circ H \circ P_T$ where $P_T f(x) = f(x)\,\mathbf{1}_{[0,T]}(x)$ is a *time–limiting operator*, $\widehat{Q_W f}(\nu) = \widehat{f}(\nu)\,\mathbf{1}_{[-\frac{W}{2},\frac{W}{2}]}(\nu)$ is a *frequency–limiting operator*, and the spreading function η_H of H is supported on A. The spreading function $\eta_{H_{W,T}}$ of $H_{W,T}$ is therefore not compactly supported and in particular not restricted to A. For his class of operators, Bello was able to prove necessity of $S_A \leq 1$ for identifiability, and to reduce the sufficiency condition of $S_A < 1$ to linear algebra, that is, to the invertibility of a matrix of a finite number of time and frequency shifts of a prototype vector. Bello gave heuristic arguments for the existence of a prototype vector which guarantees the invertibility of this matrix. His assertion has been proven only recently [9]. It is worth noting that the same prototype vectors play a crucial role in this paper.

Using an approach similar to Bello's together with more novel techniques from Gabor analysis we shall give a complete proof of both of Bello's assertions. Our approach does not require a time–gate for the input signal and a frequency–gate for the output signal. In fact, letting $\mathrm{vol}^-(M)$ and $\mathrm{vol}^+(M)$ denote the inner and outer Jordan content of M, respectively, (see (9) and (10)), we prove the following theorem:

Theorem 1.1: $\mathcal{H}_M$ is identifiable if $\mathrm{vol}^+(M) < 1$, and not identifiable if $\mathrm{vol}^-(M) > 1$.

Here, $\mathcal{H}_M$ denotes a class of Hilbert–Schmidt operators with the property that their time–frequency spread is contained in the set M, that is, those operators H with $\eta_H(t,\nu) = 0$ for $(t,\nu) \notin M$.

The operator classes discussed in this paper are relevant not only to communications engineering. In fact, the work of Kailath and Bello was greatly influenced by the work of Green and Price on radar measurements [10]–[12]. See [7] for remarks on radar and other applications.

A comparison of our result to Heisenberg's uncertainty principle is described in [7], in particular, we would like to point to connections with minimal rectangles in phase space as described in [13].

The mathematical framework used in this paper is described in Section II. In Section III, we shall prove that if the set M satisfies $\mathrm{vol}^+(M) < 1$ then the corresponding operator class allows identification, and in Section IV, we shall prove that if $\mathrm{vol}^-(M) > 1$, then the corresponding operator class does not allow identification. Note that if $\mathrm{vol}^+(M) = \mathrm{vol}^-(M) = \mathrm{vol}(M)$ then M is Jordan measurable and $\mathrm{vol}(M) = S_M$ equals the Lebesgue measure of M (see Section II-E). Moreover, a set that is not Jordan measurable is pathological in some sense (see Proposition 2.2). Consequently any spreading support set likely to arise in engineering practice will be Jordan measurable.

II. Preliminaries

In Section II-A we motivate the use of Gabor analysis as a natural tool in examining properties of operators related to their spreading functions, and present some basic results in that theory that are used in this paper. In Section II-B we discuss the principles of channel measurement and operator identification. We describe our choice of domain space X and target space Y in Section II-C and the operator spaces $\mathcal{H}_M$ in Section II-D. In Section II-E we discuss Jordan domains and the inner and outer Jordan content of sets in euclidean space. These concepts will be used to describe spreading supports and their sizes. In Section II-F, we present some results from the theory of finite Gabor frames that are used in the proof of Theorem 3.1.

Throughout this paper we are using the notation of [14] and [7].

A. Gabor analysis on $L^2(\mathbb{R}^d)$

One of the fundamental tools used in this paper is Gabor theory. This is natural in light of the fact that the operator models of communication channels we consider can be realized as superpositions on time shifts and frequency shifts (modulations) as seen in (3). Below we will describe some of the basic notions from Gabor theory that are used in this paper and indicate why they arise naturally in these investigations.

A *Hilbert–Schmidt operator* H is a bounded linear operator on $L^2(\mathbb{R}^d)^2$ which can be represented as an integral operator

$$\begin{aligned} Hf(x) &= \int \kappa_H(x,t)f(t)\,dt \\ &= \int \kappa_H(x,x-t)f(x-t)\,dt \quad (a.e.), \end{aligned} \tag{1}$$

with kernel $\kappa_H \in L^2(\mathbb{R}^2)$ [15], [16]. The linear space of Hilbert–Schmidt operators $HS(L^2(\mathbb{R}^d))$ is endowed with the Hilbert space structure of $L^2(\mathbb{R}^2)$ by setting

$$\langle H_1, H_2\rangle_{\mathrm{HS}} = \langle \kappa_{H_1}, \kappa_{H_2}\rangle_{L^2}.$$

The spreading function η_H of a Hilbert–Schmidt operator H is given by

$$\eta_H(t,\nu) = \int \kappa_H(x,x-t)e^{-2\pi i\nu x}\,dx \quad (a.e.) \tag{2}$$

and leads to a representation of H as an operator valued integral by means of

$$H = \iint \eta_H(t,\nu)\,T_tM_\nu\,dt\,d\nu\,. \tag{3}$$

Note that

$$\|\eta_H\|_{L^2} = \|\kappa_H\|_{L^2} = \|H\|_{HS}.$$

Equation (3) illustrates that support restrictions on η_H reflect limitations on the maximal time and frequency shifts which the input signals undergo: Hf is a continuous superposition of time–frequency shifted versions of f with weighting function η_H.

Further, note that if an operator H satisfies $\operatorname{supp}\eta_H(\cdot,\nu) \subseteq [0,a]$ for all $\nu \in \widehat{\mathbb{R}}$, then $\kappa_H(x,x-t)$ vanishes for $x \in \mathbb{R}$ and $t \notin [0,a]$, and for f with $\operatorname{supp} f \subseteq [0,T]$ we have $\operatorname{supp} Hf \subseteq [0,T+a]$. Similarly, if $\operatorname{supp}\eta_H(t,\cdot) \subseteq [-\frac{b}{2},\frac{b}{2}]$ for all $t \in \mathbb{R}$, then for f with $\operatorname{supp}\widehat{f} \subseteq [-\Omega,\Omega]$ we have $\operatorname{supp}\widehat{Hf} \subseteq [-(\Omega+\frac{b}{2}),\Omega+\frac{b}{2}]$. Hence, the condition

$$\operatorname{supp}\eta_H \subseteq Q_{a,b} = [0,a]\times[-\tfrac{b}{2},\tfrac{b}{2}] \tag{4}$$

for some $a,b>0$, reflects a limitation on the maximal time delay a and the maximal frequency spread $\frac{b}{2}$ produced by H. An operator which satisfies (4) for $a,b>0$ is called *underspread* if $ab \le 1$ and *overspread* if $ab>1$.

A comparison of (1) to a time–invariant convolution operator K given by $Kf(x) = \int \kappa_K(t)f(x-t)\,dt$ — whose kernel κ_K is independent of the time variable x — together with (2) shows that the condition $\eta_H(t,\cdot) \subseteq [-\frac{b}{2},\frac{b}{2}]$ for all $t\in\mathbb{R}$, excludes high frequencies and therefore rapid change of $\kappa(x,x-t)$ as a function of x. This further illuminates the role of underspread and overspread operators in the analysis of *slowly time–varying* communications channels.

The previous paragraphs emphasize the usefulness of η_H in the time–frequency analysis of operators. Additional remarks on the use of Hilbert–Schmidt operators as model of physical time–varying linear systems, as they appear in radar and in mobile communications can be found in [7], [17], [18].

[2] $L^2(\mathbb{R})$ is the Hilbert space of complex valued, square–integrable functions, that is, $f \in L^2(\mathbb{R})$ provided that $\|f\|_2^2 = \int_{\mathbb{R}} |f(x)|^2\,dx < \infty$.

Equation (3) implies that for given functions $f,\, g \in L^2(\mathbb{R}^d)$,

$$\begin{aligned} \langle Hg, f\rangle &= \iiint \eta_H(t,\nu)T_tM_\nu g(x)\overline{f(x)}\,dt\,d\nu\ dx \\ &= \iint \eta_H(t,\nu)\overline{\int f(x)\overline{T_tM_\nu g(x)}\,dx}\ dt\,d\nu \\ &= \langle \eta_H, V_gf\rangle \end{aligned} \tag{5}$$

where $V_gf(t,\nu) = \langle f, T_tM_\nu g\rangle$, $t\in\mathbb{R}^d$ and $\nu\in\widehat{\mathbb{R}}^d$ is the *short-time Fourier transform (STFT)* of f with respect to the *window function* g. It is clear then that the STFT is a natural tool to study the connection between the properties of the operator H and its spreading function η_H. If $\|g\|_{L^2(\mathbb{R}^d)} = 1$ then the STFT is an isometric isomorphism of $L^2(\mathbb{R}^d)$ onto a closed subspace of $L^2(\mathbb{R}^d\times\widehat{\mathbb{R}}^d)$. In this case the function f can be recovered by

$$f(x) = \iint V_gf(t,\nu)\,T_t\,M_\nu g(x)\,dt\,d\nu$$

whenever the integral makes sense.

One of the fundamental questions in Gabor theory is to show when a function $f\in L^2(\mathbb{R}^d)$ can be stably recovered from its *Gabor coefficients* $\{\langle f, M_{kb}T_{la}g\rangle\}_{k,l\in\mathbb{Z}^d}$ (here $a,\,b>0$ and $g\in L^2(\mathbb{R}^d)$ are fixed) or whether any f can be approximated by finite linear combinations of elements of the *Gabor system* $(g,a,b) = \{M_{kb}T_{la}g\}_{k,l\in\mathbb{Z}^d}$ [14], [19]. Specifically, we ask whether the system (g,a,b) is a *frame* for $L^2(\mathbb{R}^d)$, that is, whether there exist $A,B>0$ such that for all $f\in L^2(\mathbb{R}^d)$,

$$A\|f\|_{L^2}^2 \le \sum |\langle f, M_{kb}T_{la}g\rangle|^2 \le B\|f\|_{L^2}^2\,. \tag{6}$$

If (6) holds then every $f\in L^2(\mathbb{R}^d)$ has a stable representation

$$\begin{aligned} f &= \sum_k\sum_l \langle f, M_{kb}T_{la}\gamma\rangle\, M_{kb}T_{la}g \\ &= \sum_k\sum_l \langle f, M_{kb}T_{la}\gamma\rangle\, M_{kb}T_{la}g \ \text{ in } L^2(\mathbb{R}^d) \end{aligned} \tag{7}$$

where $\gamma\in L^2(\mathbb{R}^d)$ and (γ,a,b) is called a dual frame of (g,a,b). A frame is *tight* if $A=B$ and is *exact* if it ceases to be a frame upon the removal of a single element. A frame which is not exact is also called *overcomplete*.

For each Gabor system (g,a,b) define the *analysis map* C_g on $L^2(\mathbb{R}^d)$ by

$$C_g(f) = \{\langle f, M_{kb}T_{la}g\rangle\}_{k,l\in\mathbb{Z}^d}$$

and the *synthesis map* E_g on $l_0(\mathbb{Z}^{2d})$ (the space of finite sequences on $\mathbb{Z}^{2d}$) by

$$E_g(\{c_{k,l}\}) = \sum_{k,l} c_{k,l}\,M_{kb}T_{la}g.$$

More details on time–frequency analysis with some relevance to this paper can be found in [14].

Operator–theoretic applications of Gabor theory as presented in this paper have drawn increasing interest in applied harmonic analysis, see, for example, [20]–[29].

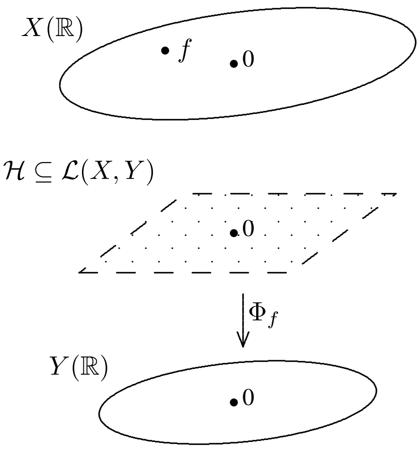

Fig. 2. Identification of an operator class $\mathcal{H}$ by a vector $f \in X(\mathbb{R})$.

B. Channel measurements

The goal of operator (or channel) identification (or measurement) is to select, for given normed linear spaces X and Y and a normed linear space of bounded linear operators $\mathcal{H} \subset \mathcal{L}(X,Y)$[3], an element $f \in X$ which induces a bounded and injective, or better, a bounded and stable linear map $\Phi_f : \mathcal{H} \longrightarrow Y,\ H \mapsto Hf$ (see Figure 2). An operator is *stable* if it is invertible and the inverse operator is bounded. Consequently, we call $\mathcal{H}$ *identifiable by* $f \in X$, if there exist $A, B > 0$ with

$$A\,\|H\|_{\mathcal{H}} \leq \|Hf\|_Y \leq B\,\|H\|_{\mathcal{H}} \tag{8}$$

for all $H \in \mathcal{H}$. Note that the fact that we only consider bounded linear operators $\mathcal{H} \subset \mathcal{L}(X,Y)$ together with $\|H\|_{\mathcal{L}(X,Y)} \leq \|H\|_{\mathcal{H}}$ guarantees that for any $f \in X$, Φ_f is bounded. Hence B in (8) always exists. Establishing identifiability is therefore equivalent to finding f so that for some positive A we have $A\,\|H\|_{\mathcal{H}} \leq \|Hf\|_Y$ for all $H \in \mathcal{H}$.

C. The Feichtinger Algebra

In this section we will describe our choice of the Banach spaces X and Y in the channel identification formalism described in Section II-B and illustrated in Figure 2.

The identification problem considered in this paper requires the use of tempered distributions such as *Dirac's delta* $\delta : f \mapsto f(0)$ and *Shah distributions* (also called *comb–functions* or *delta trains*) $\perp\!\!\!\perp\!\!\!\perp_a = \sum_{n\in\mathbb{Z}^d} \delta_{an}$, where $\delta_{na} = T_{na}\delta$ and $a > 0$, as identifiers. Hence, we have to choose a domain space $X(\mathbb{R})$ which includes some tempered distributions and, therefore, we have to deviate from a standard $L^2(\mathbb{R})$ setup.[4]

Our choice for $X(\mathbb{R})$ is the dual $S_0'(\mathbb{R})$ of Feichtinger's Banach algebra $S_0(\mathbb{R})$ which has been introduced in [30] and which has developed into a major tool in Gabor analysis.

The *Feichtinger algebra* $S_0(\mathbb{R}^d)$ is defined as follows. Let $A(\mathbb{R}^d)$ be the space of Fourier transforms of functions in $L^1(\mathbb{R}^d)$[5], with norm $\|\widehat{f}\|_A = \|f\|_{L^1}$, $l^1(\mathbb{Z}^d)$ the space of absolutely summable sequences [31] and suppose that $\psi \in A(\mathbb{R}^d)$ has compact support and satsifies $\sum_{n\in\mathbb{Z}^d} T_n\psi = 1$. Then $f \in S_0(\mathbb{R}^d)$ provided that $\sum_{n\in\mathbb{Z}^d} \|f \cdot T_n\psi\|_A < \infty$. Moreover, for each such $\psi \in A(\mathbb{R}^d)$,

$$\|f\|_{S_0} = \sum_{n\in\mathbb{Z}^d} \|f \cdot T_n\psi\|_A$$

defines an equivalent norm on $S_0(\mathbb{R}^d)$. Intuitively, $f \in S_0(\mathbb{R}^d)$ if and only if f is locally in $A(\mathbb{R}^d)$ with global decay of l^1–type. $S_0(\mathbb{R}^d)$ is therefore the same as the *Wiener amalgam space* $W(A(\mathbb{R}^d), l^1(\mathbb{Z}^d))$ (see for example, [14]).

An equivalent characterization of $S_0(\mathbb{R}^d)$ is the following.

$$S_0(\mathbb{R}^d) = \Big\{\ f \in L^2(\mathbb{R}^d) : \\ V_{g_0} f\,(t,\nu) = \int f(x)\ e^{-2\pi i \nu x} g_0(x-t)\,dx \\ \in L^1(\mathbb{R}^d \times \widehat{\mathbb{R}}^d)\Big\}$$

where $V_{g_0} f$ is the STFT of f with respect to the gaussian window $g_0(x) = e^{-\pi\|x\|_2^2}$, $x \in \mathbb{R}^d$. In fact, $\|f\| = \|V_{g_0} f\|_{L^1(\mathbb{R}^d\times\widehat{\mathbb{R}}^d)}$ is an equivalent norm on $S_0(\mathbb{R}^d)$.

The dual space $S_0'(\mathbb{R}^d)$ of the Feichtinger algebra satisfies $S_0'(\mathbb{R}^d) = W(A'(\mathbb{R}^d), l^\infty(\mathbb{Z}^d))$[32]. Intuitively, $f \in S_0'(\mathbb{R}^d)$ if and only if f is locally the Fourier transform of a bounded function (that is, in $A'(\mathbb{R}^d)$) and that these local norms are uniformly bounded. Hence, $S_0'(\mathbb{R}^d)$ contains Dirac's delta δ and Shah distributions.

Since we are considering only the one-dimensional setting in this paper, we will take $X = S_0'(\mathbb{R})$ and $Y = L^2(\mathbb{R})$.

D. Hilbert–Schmidt operators with bandlimited symbols

In this section, we describe the operator class $\mathcal{H} \subset \mathcal{L}(X,Y) = \mathcal{L}(S_0'(\mathbb{R}), L^2(\mathbb{R}))$ that appears in the channel identification formalism described in Section II-B and illustrated in Figure 2.

As mentioned in Section II-C, our results require the use of Shah distributions as identifiers. Not all Hilbert–Schmidt operators in $\mathcal{L}(L^2(\mathbb{R}), L^2(\mathbb{R}))$ can be extended to act on a space of distributions containing the Shah distribution, hence, we shall narrow the class of operators considered to those which satisfy a regularity condition on their kernels. Since for $a > 0$ $\perp\!\!\!\perp\!\!\!\perp_a \in S_0'(\mathbb{R})$, it is natural to choose

$$\mathcal{H} = \left\{H \in HS(L^2(\mathbb{R})) :\ \kappa_H \in S_0(\mathbb{R}^2)\right\},$$

since then $\mathcal{H} \subseteq \mathcal{L}(S_0'(\mathbb{R}), S_0(\mathbb{R})) \subset \mathcal{L}(S_0'(\mathbb{R}), L^2(\mathbb{R}))$ [33].

[3] For Banach spaces X and Y, $\mathcal{L}(X,Y)$ denotes the Banach space of bounded (that is, continuous) linear operators from X into Y. The norm $\|H\|_{\mathcal{L}(X,Y)}$ is the standard operator norm given by $\|H\|_{\mathcal{L}(X,Y)} = \sup_{x\in X} \|Hx\|_Y / \|x\|_X$.

[4] A possible choice for $X(\mathbb{R})$ would be the space of all tempered distributions $\mathcal{S}'(\mathbb{R})$, which is the dual of the Fréchet space of Schwartz functions $\mathcal{S}(\mathbb{R})$ and which is equipped with the weak–$*$ topology. Certainly, we would rather choose a Banach space as domain $X(\mathbb{R})$, since this would give us the convenience of expressing continuity (boundedness) and openness (stability) of linear operators by means of norm inequalities. The results in this paper are consequences of the structure of the identification problem at hand, and not of topological subtleties. Our choice to work with the Banach spaces $S_0(\mathbb{R})$ and $S_0'(\mathbb{R})$ as opposed to the Fréchet space of of Schwartz functions $\mathcal{S}(\mathbb{R}) \subset S_0(\mathbb{R})$ was made for convenience only.

[5] $L^1(\mathbb{R}^d)$ is the Banach space of complex valued, integrable functions, that is, $f \in L^1(\mathbb{R}^d)$ provided that $\|f\|_1 = \int_{\mathbb{R}} |f(x)|\,dx < \infty$.

Identifiability will be shown for operator classes with compactly supported spreading functions, that is, we consider operator classes of the form

$$\mathcal{H}_M = \{H \in \mathcal{H}: \ \operatorname{supp} \eta_H \subseteq M\}\,, \quad M \subset \mathbb{R}\times\widehat{\mathbb{R}}.$$

Note that $\mathcal{H}_M \subseteq \mathcal{H}_{M'}$ if $M \subseteq M'$, and that the linear spaces $\mathcal{H}$ and $\mathcal{H}_M$, $M \subset \mathbb{R}\times\widehat{\mathbb{R}}$, are not closed as linear subspaces of the space of Hilbert–Schmidt operators.

E. Jordan domains and Jordan content

The stated goal of this paper is to extend the proof of Kailath's conjecture which is proved for rectangles and parallelograms in [7] to "essentially arbitrary" regions. In this section we describe more precisely what is meant by "essentially arbitrary" from a mathematical point of view. Taking into account the requirements of the proof of Theorem 3.1, we are led naturally to the notion of Jordan content and Jordan domains. The definition we use for convenience differs from but is equivalent to those found in most textbooks, for example, see [34].

Definition 2.1: For $K, L \in \mathbb{N}$ set $R_{K,L} = [0, \frac{1}{K}] \times [0, \frac{K}{L}]$ and

$$\mathcal{U}_{K,L} = \left\{ \bigcup_{j=1}^{J} \Big(R_{K,L} + (\tfrac{k_j}{K}, \tfrac{p_j K}{L}) \Big) : \ k_j, p_j \in \mathbb{Z}, J \in \mathbb{N} \right\}.$$

Let $M \subset \mathbb{R}\times\widehat{\mathbb{R}}$ be bounded and let μ be the Lebesgue measure on $\mathbb{R}\times\widehat{\mathbb{R}}$. The *inner content* of M is defined as

$$\operatorname{vol}^{-}(M) = \sup\{\mu(U) : U{\subset}M \text{ and } U{\in}\mathcal{U}_{K,L} \text{ for some } K,L{\in}\mathbb{N}\} \quad (9)$$

and the *outer content* of M is given by

$$\operatorname{vol}^{+}(M) = \inf\{\mu(U) : U{\supset}M \text{ and } U{\in}\mathcal{U}_{K,L} \text{ for some } K,L{\in}\mathbb{N}\} \quad (10)$$

Clearly, we have $\operatorname{vol}^{-}(M) \leq \operatorname{vol}^{+}(M)$ and if $\operatorname{vol}^{-}(M) = \operatorname{vol}^{+}(M)$, then we say that M is a *Jordan domain* with *Jordan content* $\operatorname{vol}(M) = \operatorname{vol}^{-}(M) = \operatorname{vol}^{+}(M)$.

In the following proposition we collect some relevant facts on Jordan content (see for example, [34]).

Proposition 2.2: Let $M \subset \mathbb{R}\times\widehat{\mathbb{R}}$.

1) If M is a Jordan domain, then M is Lebesgue measurable with $\mu(M) = \operatorname{vol}(M)$.
2) If M is Lebesgue measurable and bounded and its boundary ∂M is a Lebesgue zero set, that is, $\mu(\partial M) = 0$, then M is a Jordan domain.
3) If M is open, then $\operatorname{vol}^{-}(M) = \mu(M)$ and if M is compact, then $\operatorname{vol}^{+}(M) = \mu(M)$.
4) If $\mathcal{P} \subset \mathbb{N}$ is unbounded, then replacing the quantifier "*for some* $L \in \mathbb{N}$" with "*for some* $L \in \mathcal{P}$" in (9) and in (10) leads to equivalent definitions of inner and outer Jordan content.

Proposition 2.2(1)–(3) suggest that the class of sets considered in our results is very broad. Bounded sets that are Jordan measurable but not Lebesgue measurable are pathological in some sense and hence unlikely to arise as the spreading support of a physically realistic communication channel. Proposition 2.2(4) will be used in the proof of Theorem 3.1. The initial step in that proof is to approximate the spreading support of our channel operator by a union of boxes $U \in \mathcal{U}_{K,L}$ for some K, $L \in \mathbb{N}$. For technical reasons (see Proposition 2.3) we must choose L to be a prime number and $K \leq L$.

F. Gabor analysis on $\mathbb{C}^L$

Discrete Gabor systems on finite dimensional spaces can be defined in a natural way and properties of such systems will be used in Section III. Let $L \in \mathbb{N}$ be fixed and let $\omega = e^{-2\pi i/L}$. The translation operator T is the unitary operator on $\mathbb{C}^L$ given by $Tx = T(x_0, \ldots, x_{L-1}) = (x_{L-1}, x_0, x_1, \ldots, x_{L-2})$, and the modulation operator M is the unitary operator given by $Mx = M(x_0, \ldots, x_{L-1}) = (\omega^0 x_0, \omega^1 x_1, \ldots, \omega^{L-1} x_{L-1})$. Given a vector $c \in \mathbb{C}^L$ the *full Gabor system with window* c is the collection $\{M^l T^k c\}_{l,k=0}^{L-1}$.

A more concrete representation of the full Gabor system is obtained by realizing the elements of the system as columns in a matrix. Given $L \in \mathbb{N}$, we define the *discrete* $L \times L$ *Fourier matrix* W_L by $W_L = (\omega^{pq})_{p,q=0}^{L-1}$. Let $c = (c_0, c_1, \ldots, c_{L-1}) \in \mathbb{C}^L$ be given and for $k = 0, 1, \ldots, L-1$, let D_k be the diagonal matrix

$$D_k = \begin{pmatrix} c_k & & & & & \\ & c_{k+1} & & & & \\ & & \ddots & & & \\ & & & c_{L-1} & & \\ & & & & c_0 & \\ & & & & & \ddots \\ & & & & & & c_{k-1} \end{pmatrix}.$$

Define the $L \times L^2$ *full Gabor system matrix* by

$$(D_0{\cdot}W_L \,|\, D_1{\cdot}W_L \,|\, \cdots \,|\, D_{L-1}{\cdot}W_L) \equiv (A_0 \,|\, A_1 \,|\, \cdots \,|\, A_{L-1}). \quad (11)$$

In fact,

$$A_k = (c_{p+k}\, \omega^{qp})_{p,q=0}^{L-1}$$

where the subscript of c is taken modulo L. It is clear that the columns of the matrix (11) are the vectors $\{M^l T^k c\}_{l,k=0}^{L-1}$. We will see that the matrix (11) arises naturally in the proof of Theorem 3.1. The fundamental property of the full Gabor system matrix is the following.

Proposition 2.3: If L is prime then for generic[6] choices of $c \in \mathbb{C}^L$, every subset of L columns of the full Gabor system matrix is linearly independent in $\mathbb{C}^L$.

III. Sufficiency of $\operatorname{vol}(M) < 1$ for the identifiability of $\mathcal{H}_M$

In this section, we shall prove the following theorem.

Theorem 3.1: The class $\mathcal{H}_M$ is identifiable if $\operatorname{vol}^{+}(M){<}1$.

[6] Here the term "generic" means that the set of such c is a dense open set of full Lebesgue measure in $\mathbb{C}^L$. In particular this means that the conclusion of Proposition 2.3 holds for almost every c in the sense of Lebesgue measure.

A. Summary of the proof of Theorem 3.1

The case that M is a rectangle, that is, $M = [a_1, a_2]\times[b_1, b_2]$ for some $a_2 > a_1 > 0$ and $b_2 > b_1 > 0$, has been considered by Kailath [1], [2]. If $\mathrm{vol}^+(M) = (a_2 - a_1)(b_2 - b_1) \le 1$, then $\perp\!\!\perp\!\!\perp_a$ identifies $\mathcal{H}_M$ whenever $a_2 - a_1 \le a \le (b_2 - b_1)^{-1}$, since $H\perp\!\!\perp\!\!\perp_a$ records samples of $\kappa_H(\cdot, \cdot - t)$ at least at the critical sampling rate $(b_2 - b_1)^{-1}$, while $a \ge a_2 - a_1$ guarantees that no aliasing of samples takes place. See Figure 3 for details. The situation for M not being contained in a rectangle of volume at most one is more complicated (see Figure 4).

In order to give an intuition of the proof, assume that the support of η_H is contained in the rectangle $[0, 1] \times [0, K]$. This means that for each $t \in [0, 1]$ the function $\kappa_H(\cdot, \cdot - t)$, which is simply $\kappa_H(x, y)$ restricted to the line $y = x - t$, is bandlimited to the interval $[0, K]$. It is therefore sufficient to obtain samples of the function $\kappa_H(\cdot, \cdot - t)$ at the points $\frac{k}{K}$, for $k \in \mathbb{Z}$.

In essence, a similar approach to the proof for rectangles (see [7]) allows us to obtain weighted sums of samples of $\kappa_H(\cdot, \cdot - t)$ rather than the samples themselves. By employing a technique reminiscent of that used in Papoulis' generalized sampling we are able to reduce the problem to a finite linear system of L equations in KL unknowns. While in general unsolvable, we observe that by our assumptions on the support size of η_H, all but L of the unknowns are zero.

It remains only to establish that the $L \times L$ system we arrive at admits a solution. The norm inequality (8) now follows from standard considerations from linear algebra.

We divide the proof of Theorem 3.1 into four parts. In Section III-B we slightly simplify our setup. For fixed M, we construct in Section III-C a distribution $f \in S_0'(\mathbb{R})$ which will be used for identification later in the proof. This f induces a linear map

$$\Phi_f : \mathcal{H}_M \longrightarrow L^2(\mathbb{R}), \;\; H \mapsto Hf$$

whose injectiveness is shown in Section III-D. In Section III-E we shall see that that Φ_f is indeed bounded and stable, implying, finally, that $\mathcal{H}_M$ is identifiable by f.

B. Assume $M \subseteq [0, 1] \times [0, \infty)$

As the first step in the proof, we assume without loss of generality that if $(t, \nu) \in M$ then $t \in [0, 1]$ and $\nu \ge 0$. To see why this can be done, suppose that for some $a, b > 0$, $M \subseteq [0, a] \times [-\frac{b}{2}, \frac{b}{2}]$ and $H \in \mathcal{H}_M$. Define $\widetilde{H} \in \mathcal{H}$ via

$$\kappa_{\widetilde{H}}(x, y) = a\, \kappa_H(ax, ay)\, e^{\pi i abx}.$$

Then it is easy to see that

$$\begin{aligned} \eta_{\widetilde{H}}(t, \nu) &= \int \kappa_{\widetilde{H}}(x, x - t)\, e^{-2\pi i \nu x}\, dx \\ &= \int a\kappa_H(ax, a(x - t))\, e^{\pi i abx}\, e^{-2\pi i \nu x}\, dx \\ &= \int \kappa_H(x, x - at)\, e^{-2\pi i (\frac{\nu}{a} - \frac{b}{2})x}\, dx \\ &= \eta_H(at, \tfrac{\nu}{a} - \tfrac{b}{2}) \end{aligned}$$

and that if $\mathrm{supp}\, \eta_H \subseteq [0, a]\times[-\frac{b}{2}, \frac{b}{2}]$ then $\mathrm{supp}\, \eta_{\widetilde{H}} \subseteq [0, 1] \times [0, ab]$.

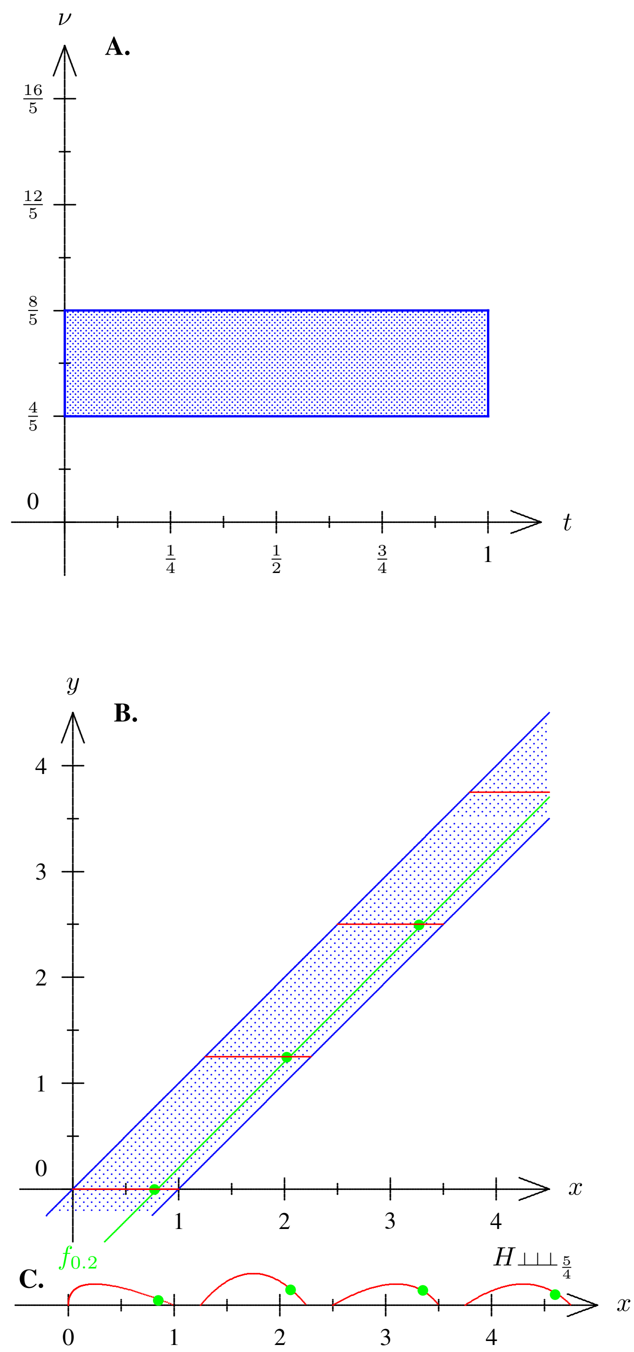

Fig. 3. Identification of $\mathcal{H}_M$ for $M = [\frac{4}{5}, \frac{8}{5}]\times[0, 1] \in \mathcal{U}_{4,5}$. **A.** Spreading support set M, vol $M = \frac{4}{5} < 1$. **B.** Support of kernel κ_H of $H \in \mathcal{H}_M$. The function κ_H is bandlimited along the diagonals, that is, $\mathrm{supp}\,\widehat{\kappa_H(\cdot, \cdot - t)} = \mathrm{supp}\,\eta_H(\cdot, t) \subseteq [\frac{4}{5}, \frac{8}{5}]$ for all $t \in [0, 1]$. Here, $f_{0.2}(x) = \kappa_H(x, x - (1 - 0.2))$, $x \in \mathbb{R}$. **C.** Channel output $H\perp\!\!\perp\!\!\perp_{\frac{5}{4}}$, which contains all sampling values of κ_H needed to reconstruct κ_H and therefore H. Sampling values of $f_{0.2}$ are singled out.

C. Construction of the identifier f

The identifying distribution will have the form

$$f = \sum_k c_k \, \delta_{\frac{k}{K}} \tag{12}$$

for some $K \in \mathbb{N}$ and some $c = (c_0, c_1, \ldots, c_{L-1}) \in \mathbb{C}^L$, where indices of c_k in the sum are taken modulo L. The goal is to choose K, L, and c so that the aliasing in Figure 4.**C** can be controlled in such a way that κ_H and therefore H can be recovered from Hf. Below we will determine appropriate parameters K, L, and c needed to define f.

Assume that $\mathrm{vol}^+(M) < 1$. With $\mathcal{P}$ denoting the set of prime numbers, Proposition 2.2(4) says that we can choose $K, L \in \mathbb{N}$ with L prime so that *(i)* $M \subseteq [0,1] \times [0,K]$, *(ii)* $L \geq K$, and *(iii)* $M \subseteq U_M$ with $U_M \in \mathcal{U}_{K,L}$ and $\mathrm{vol}\,(U_M) \leq 1$, that is,

$$U_M = \bigcup_{j=0}^{J-1} \left(R_{K,L} + (\tfrac{k_j}{K}, \tfrac{p_j K}{L}) \right), \qquad k_j, p_j \in \mathbb{Z}, J \in \mathbb{N},$$

where $R_{K,L} = [0, \frac{1}{K}] \times [0, \frac{K}{L}]$ and where $(k_j, p_j) \neq (k_{j'}, p_{j'})$ if $j \neq j'$. Note further that $1 \geq \mathrm{vol}\,(U_M) = J\,\mathrm{vol}\,(R_{K,L}) = \frac{J}{L}$ implies $J \leq L$. Since $\mathcal{H}_M \supset \mathcal{H}_{M'}$ if $M \supset M'$, the identifiability of $\mathcal{H}_M$ implies the identifiability of $\mathcal{H}_M \supset \mathcal{H}_{M'}$, and, by adding some additional cells to M if necessary, we can assume in the remainder of this section that $J = L$.

For any $c = (c_0, c_1, \ldots, c_{L-1}) \in \mathbb{C}^L$, let $A(c)$ denote the $L \times KL$ matrix

$$A(c) = [A_0 \;\; A_1 \;\; \cdots \;\; A_{K-1}] \tag{13}$$

where the $L \times L$ matrices A_k are defined by (11) and have the form

$$A_k = (c_{p+k}\, \omega^{qp})_{p,q=0}^{L-1}$$

where $\omega = e^{-2\pi i/L}$ and where the subscripts on c are taken modulo L. Since $K \leq L$, the matrix A is a submatrix of the full Gabor system matrix (11). In light of Proposition 2.3 there exists $c \in \mathbb{C}^L$ so that every $L \times L$ submatrix of $A(c)$ is invertible.

Choose such a c and define f as in (12).

D. Determining $H \in \mathcal{H}_M$ from $H(f)$.

The operator $H \in \mathcal{H}_M$ is completely determined by its kernel κ_H. Therefore it is sufficient to show that κ_H can be recovered from $H(f)$.

For convenience in the calculations that follow, we define for $t \in [0, \frac{1}{K}]$ and $0 \leq k < K$ the function

$$\kappa_k(t, x) = \kappa_H(1 - t - \tfrac{k}{K} + x, x).$$

This means that the kernel $\kappa_H(x, x-t)$ can be recovered via

$$\kappa_H(x, x-t) = \sum_{k=0}^{K-1} \kappa_k(1 - t - \tfrac{k}{K}, x - t)$$

if we assume that $\kappa_k(t,x) = 0$ for $t \notin [0, \frac{1}{K}]$.

Since $H \in \mathcal{H}_M$ with $M \subseteq [0,1] \times [0,K]$ means that $\mathrm{supp}\,\eta_H \subseteq [0,1] \times [0,K]$ with η_H given by (2), it follows

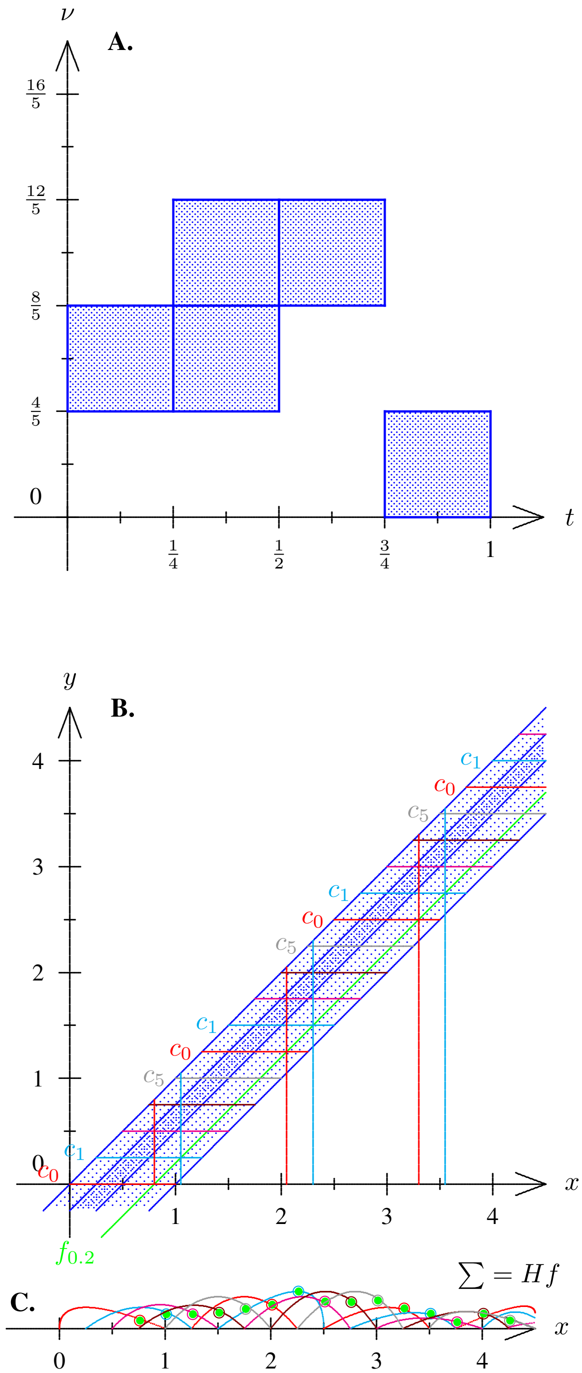

Fig. 4. Identification of $\mathcal{H}_M$ for $M \in \mathcal{U}_{4,5}$ not being a rectangle. **A.** Spreading support set M, vol $M = 1$. **B.** Support of kernel κ_H of $H \in \mathcal{H}_M$. The bandlimitation of $\kappa_H(\cdot, \cdot - t)$ along the diagonals depends on t. **C.** The channel output $H(\sum_k c_k \delta_{\frac{k}{4}})$ is the sum over all functions displayed here, leading to aliasing of samples in the channel output $H(\sum_k c_k \delta_{\frac{k}{4}})$. Samples of $f_{0.2}$ contributing to the weighted sum are marked.

that $\kappa_k(t,\cdot)$ is bandlimited to $[0,K]$ for each t and k. The Fourier transform of $\kappa_k(t,x)$ in the second variable is

$$e^{2\pi i\nu(1-t-\frac{k}{K})}\,\eta_H(1-t-\tfrac{k}{K},\nu)\equiv\eta_k(t,\nu).$$

Therefore η_H and subsequently κ_H is completely determined by $\eta_k(t,\nu+\frac{pK}{L})$ for $(t,\nu)\in R_{K,L}$, $0\le k<K$ and $0\le p<L$. In particular

$$\eta_H(t,\nu+\tfrac{pK}{L})=e^{-2\pi i\nu t}\sum_{k=0}^{K-1}\eta_k(1-t-\tfrac{k}{K},\nu+\tfrac{pK}{L})$$

for $(t,\nu)\in R_{K,L}$ if we assume $\eta_k(t,\nu)=0$ for $t\notin[0,K]$.

With f as in (12),

$$(Hf)(x)=\big\langle\kappa(x,\cdot),\sum_k c_k\delta_{\frac{k}{K}}(\cdot)\big\rangle=\sum_k c_k\,\kappa_H(x,\tfrac{k}{K}).$$

As previously mentioned, in this sum and subsequently, all subscripts of c are taken modulo L. For $t\in[0,\frac{1}{K}]$ let $x_n(t)=(1-t)+\frac{n}{K}$. Because $\operatorname{supp}\kappa_H(x,y)$ is contained in the strip between the line $y=x$ and $y=x-1$, it follows that for each fixed x, $\kappa_H(x,y)$ vanishes for y outside $[x-1,x]$ (see Figure 4). Hence $\kappa_H(x_n(t),\frac{k}{K})\neq 0$ only if $n\le k<n+K$, $(Hf)(x_n(t))$ reduces to a finite sum and we write

$$\begin{aligned}
s_n(t)\ &\equiv\ (Hf)(x_n(t)) &(14)\\
&=\ \sum_k c_k\,\kappa_H(x_n(t),\tfrac{k}{K})\\
&=\ \sum_{k=0}^{K-1}c_{n+k}\,\kappa_H(1-t+\tfrac{n}{K},\tfrac{n+k}{K})\\
&=\ \sum_{k=0}^{K-1}c_{n+k}\,\kappa_H(1-t-\tfrac{k}{K}+\tfrac{n+k}{K},\tfrac{n+k}{K})\\
&=\ \sum_{k=0}^{K-1}c_{n+k}\,\kappa_k(t,\tfrac{n+k}{K}).
\end{aligned}$$

Letting $n=mL+p$ with $m\in\mathbb{Z}$ and $0\le p<L$, we write $s_m^p(t)=s_{mL+p}(t)$. Then

$$\begin{aligned}
s_m^p(t)\ &=\ \sum_{k=0}^{K-1}c_{mL+p+k}\,\kappa_k\left(t,\tfrac{mL}{K}+\tfrac{p+k}{K}\right)\\
&=\ \sum_{k=0}^{K-1}c_{p+k}\,\kappa_k\left(t,\tfrac{p+k}{K}+\tfrac{mL}{K}\right). &(15)
\end{aligned}$$

For each $0\le p<L$ form the Fourier series

$$\begin{aligned}
&G_p(t,\nu)\\
&\equiv\ \tfrac{L}{K}\sum_m s_m^p(t)\,e^{-2\pi i\nu mL/K}\\
&=\ \tfrac{L}{K}\sum_m\sum_{k=0}^{K-1}c_{p+k}\,\kappa_k\left(t,\tfrac{p+k}{K}+\tfrac{mL}{K}\right)e^{-2\pi i\nu mL/K}\\
&=\ \tfrac{L}{K}\sum_{k=0}^{K-1}c_{p+k}\sum_m\kappa_k\left(t,\tfrac{p+k}{K}+\tfrac{mL}{K}\right)e^{-2\pi i\nu mL/K}\\
&=\ \sum_{k=0}^{K-1}c_{p+k}\sum_q\eta_k\left(t,\nu+\tfrac{qK}{L}\right)e^{-2\pi i\left(\frac{p+k}{K}\right)\left(\nu+\frac{qK}{L}\right)}
\end{aligned}$$

where we have applied the Poisson Summation Formula on the last line. Assuming that $\nu\in[0,\frac{K}{L}]$ and since $\eta_k(t,\cdot)$ is supported in the interval $[0,K]$, it follows that the above sum over q is finite and therefore for each $0\le p<L$ we have

$$G_p(t,\nu)=\sum_{k=0}^{K-1}\sum_{q=0}^{L-1}c_{p+k}\,\eta_k\left(t,\nu+\tfrac{qK}{L}\right)e^{-2\pi i\left(\frac{p+k}{K}\right)\left(\nu+\frac{qK}{L}\right)}.$$

Manipulating this expression, we arrive at the streamlined system

$$\widetilde{G}_p(t,\nu)=\sum_{k=0}^{K-1}\sum_{q=0}^{L-1}c_{p+k}\,e^{-2\pi ipq/L}\,\widetilde{\eta}_k\left(t,\nu+\tfrac{qK}{L}\right)\qquad(16)$$

where $\widetilde{G}_p(t,\nu)=G_p(t,\nu)\,e^{2\pi ip\nu/K}$ and $\widetilde{\eta}_k(t,\nu)=\eta_k(t,\nu)\,e^{-2\pi ik\nu/K}$. In other words, we can derive a system of L equations in KL unknowns for the functions

$$\Big\{\widetilde{\eta}_k\left(t,\nu+\tfrac{qK}{L}\right)\ :\ 0\le k<K,\ 0\le q<L,(t,\nu)\in[0,\tfrac{1}{K}]\times[0,\tfrac{K}{L}]\Big\},\ (17)$$

in which the coefficients in the equation do not depend on (t,ν). It is clear that the matrix for this system is $A(c)$ (13), and that the set of functions in (17) completely determine κ_H.

Finally, note that since $M\subseteq U_M$, and for $(t,\nu)\in R_{K,L}$, we have $\widetilde{\eta}_k\left(t,\nu+\frac{qK}{L}\right)=0$ unless $(k,q)=(k_j,p_j)$ for some $0\le j\le J-1=L-1$. Therefore (16) has no more than L nonzero terms in the double sum on the right hand side, and (16) reduces to a system of L equations in L unknowns. The matrix for this reduced system is simply a choice of L columns of the matrix $A(c)$, specifically the j^{th} column of this matrix is the k_j^{th} column of the $L\times L$ matrix A_{p_j}. Call this matrix $\mathbf{A}_M$. Our choice of c guarantees that that $\mathbf{A}_M$ is invertible.

Define the $\mathbb{C}^L$–valued functions $\boldsymbol{\eta}(t,\nu)$ and $\mathbf{G}(t,\nu)$ on $R_{K,L}$ by

$$\boldsymbol{\eta}(t,\nu)=\left(\widetilde{\eta}_{m_j}\left(t,\nu+\tfrac{n_jK}{L}\right)\right)_{j=0}^{L-1}$$

and

$$\mathbf{G}(t,\nu)=\left(\widetilde{G}_p(t,\nu)\right)_{p=0}^{L-1}.$$

The system (16) can therefore be written

$$\mathbf{A}_M\,\boldsymbol{\eta}(t,\nu)=\mathbf{G}(t,\nu),\quad(t,\nu)\in R_{K,L}.\qquad(18)$$

Since $\mathbf{A}_M$ is invertible, we can recover $\boldsymbol{\eta}$ pointwise from $\mathbf{G}$ which depends only on the channel output Hf. From $\boldsymbol{\eta}$ we can recover η_H and hence the kernel $\kappa_H(x,x-t)$ of H.

E. Boundedness and stability

Here we show that (8) holds.

Lemma 3.2: With f given in Section III-C, $H\in\mathcal{H}_M$, and $\boldsymbol{\eta}$ and $\mathbf{G}$ as in (18),

(a) $\|H(f)\|^2_{L^2(\mathbb{R})}=\iint_{R_{K,L}}\|\mathbf{G}(t,\nu)\|^2_{\mathbb{C}^L}\,dt\,d\nu.$

(b) $\|H\|^2_{\mathcal{H}}=\|\kappa_H\|^2_{L^2(\mathbb{R}^2)}=\iint_{R_{K,L}}\|\boldsymbol{\eta}(t,\nu)\|^2_{\mathbb{C}^L}\,d\nu\,dt.$

Proof: (a) Using the definition of s_n in (14) and the definition of s_m^p in (15), we have

$$\begin{aligned}
\|H(f)\|^2_{L^2(\mathbb{R})} &= \int_{-\infty}^{\infty} |Hf(t)|^2\,dt \\
&= \int_{-\infty}^{\infty} |Hf(1-t)|^2\,dt \\
&= \sum_n \int_{\frac{n}{K}}^{\frac{n+1}{K}} |Hf(1-t)|^2\,dt \\
&= \sum_n \int_0^{\frac{1}{K}} |Hf(1-t-\tfrac{n}{K})|^2\,dt \\
&= \sum_n \int_0^{\frac{1}{K}} |Hf(x_n(t))|^2\,dt \\
&= \sum_n \int_0^{\frac{1}{K}} |s_n(t)|^2\,dt \\
&= \int_0^{\frac{1}{K}} \sum_{p=0}^{L-1}\sum_m |s_m^p(t)|^2\,dt \\
&= \int_0^{\frac{1}{K}} \sum_{p=0}^{L-1}\int_0^{\frac{K}{L}} |G_p(t,\nu)|^2\,dt\,d\nu \\
&= \iint_{R_{K,L}} \sum_{p=0}^{L-1} |G_p(t,\nu)|^2\,dt\,d\nu \\
&= \iint_{R_{K,L}} \sum_{p=0}^{L-1} |\widetilde{G}_p(t,\nu)|^2\,dt\,d\nu \\
&= \iint_{R_{K,L}} \|\mathbf{G}(t,\nu)\|^2_{\mathbb{C}^L}\,dt\,d\nu.
\end{aligned}$$

(b) Similarly,

$$\begin{aligned}
\int_{-\infty}^{\infty} & \int_{-\infty}^{\infty} |\kappa_H(x,y)|^2\,dx\,dy \\
&= \int_0^1\int_{-\infty}^{\infty} |\kappa_H(x,x-t)|^2\,dx\,dt \\
&= \int_0^1\int_{-\infty}^{\infty} |\kappa_H(1-t+x,x)|^2\,dx\,dt \\
&= \sum_{k=0}^{K-1}\int_0^{\frac{1}{K}}\int_{-\infty}^{\infty} |\kappa_H(1-t+x,x)|^2\,dx\,dt \\
&= \sum_{k=0}^{K-1}\int_0^{\frac{1}{K}}\int_{-\infty}^{\infty} |\kappa_k(t,x)|^2\,dx\,dt \\
&= \sum_{k=0}^{K-1}\int_0^{\frac{1}{K}}\int_{-\infty}^{\infty} |\eta_k(t,\nu)|^2\,d\nu\,dt \\
&= \sum_{k=0}^{K-1}\int_0^{\frac{1}{K}}\int_0^{K} |\eta_k(t,\nu)|^2\,d\nu\,dt \\
&= \sum_{k=0}^{K-1}\sum_{q=0}^{L-1}\int_0^{\frac{1}{K}}\int_0^{\frac{K}{L}} |\eta_k(t,\nu+q\tfrac{K}{L})|^2\,d\nu\,dt \\
&= \iint_{R_{K,L}} \sum_{k=0}^{K-1}\sum_{q=0}^{L-1} |\eta_k(t,\nu+q\tfrac{K}{L})|^2\,d\nu\,dt \\
&= \iint_{R_{K,L}} \sum_{j=0}^{L-1} |\eta_{k_j}(t,\nu+p_j\tfrac{K}{L})|^2\,d\nu\,dt \\
&= \iint_{R_{K,L}} \sum_{j=0}^{L-1} \|\boldsymbol{\eta}(t,\nu)\|^2_{\mathbb{C}^L}\,d\nu\,dt
\end{aligned}$$

since $\eta_k(t,\nu+q\frac{K}{L})=0$ unless $(k,q)=(k_j,p_j)$. ■

It is now clear that (8) holds by observing that by construction the matrix $\mathbf{A}_M$ of (18) is invertible and independent of $(t,\nu)\in R_{K,L}$. Hence, for $(t,\nu)\in R_{K,L}$, we have

$$\frac{1}{\|\mathbf{A}_M^{-1}\|^2}\,\|\boldsymbol{\eta}(t,\nu)\|^2_{\mathbb{C}^L} \le \|\mathbf{H}(t,\nu)\|^2_{\mathbb{C}^L} \le \|\mathbf{A}_M\|^2\,\|\boldsymbol{\eta}(t,\nu)\|^2_{\mathbb{C}^L}$$

where $\|\cdot\|$ is the Frobenius norm of a matrix, that is, the operator norm of the matrix considered as an operator on $l^2(\mathbb{Z}_L)$. Integrating this inequality over $R_{K,L}$ and applying Lemma 3.2 we obtain

$$\frac{1}{\|\mathbf{A}_M^{-1}\|}\,\|H\|_{\mathcal{H}} \le \|H(f)\|_{L^2(\mathbb{R})} \le \|\mathbf{A}_M\|\,\|H\|_{\mathcal{H}}$$

which is (8).

IV. NECESSITY OF $\mathrm{vol}\,(M) < 1$ FOR THE IDENTIFIABILITY OF $\mathcal{H}_M$

The goal of this section is to prove the following theorem.

Theorem 4.1: The class $\mathcal{H}_M$ is not identifiable if $\mathrm{vol}^-(M) > 1$.

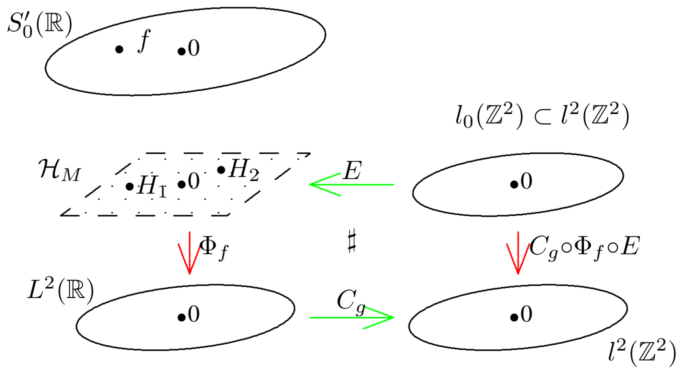

Fig. 5. Strategy for the proof that $\mathcal{H}_M$ is not identifiable if $\mathrm{vol}^-(M) > 1$. We shall show that for all $f \in S_0'(\mathbb{R})$, the bounded operator $C_g \circ \Phi_f \circ E$ is not stable. The stability of the synthesis operator E and the analysis operator C_g, together with the lack of stability of $C_g \circ \Phi_f \circ E$, shows that Φ_f is not stable.

A. Summary of the proof of Theorem 4.1

Given a bounded and measurable subset M with $\mathrm{vol}^-(M) > 1$ we show that for *every* $f \in S_0'(\mathbb{R})$, the operator

$$\Phi_f : \mathcal{H}_M \longrightarrow L^2(\mathbb{R}),\ \ H \mapsto Hf$$

is *not* stable, that is, that inequality (8) fails to hold.

As before, we assume without loss of generality that for some $K, L \in \mathbb{N}$, there is a $U \in \mathcal{U}_{K,L}$ such that $U \subseteq M$ and $\mathrm{vol}\,(U) > 1$. It will be sufficient to show that $\mathcal{H}_U$ is not identifiable since $\mathcal{H}_U \subseteq \mathcal{H}_M$. Recall that

$$U = \bigcup_{j=0}^{J-1} R_{K,L} + (\tfrac{k_j}{K}, \tfrac{p_j K}{L})$$

with $R_{K,L} = [0, \frac{1}{K}] \times [0, \frac{K}{L}]$, $0 \le k_j < K$, $0 \le p_j < L$ and $0 \le j < J \le L$.

We shall equip $l_0(\mathbb{Z}^2)$, the space of sequences on $\mathbb{Z}^2$ with only finitely many non–zero terms, with the l^2–norm and construct a bounded and stable synthesis map $E : l_0(\mathbb{Z}^2) \longrightarrow \mathcal{H}_U$, and a bounded and stable (g, a', b')–analysis operator $C_g : L^2(\mathbb{R}) \longrightarrow l^2(\mathbb{Z}^2)$ with the property that the composition

$$C_g \circ \Phi_f \circ E :\ l_0(\mathbb{Z}^2) \longrightarrow l^2(\mathbb{Z}^2), \quad f \in S_0'(\mathbb{R})$$

is not stable. Since E and C_g are stable, we have that all operators $\Phi_f : \mathcal{H}_U \longrightarrow L^2(\mathbb{R})$, $f \in S_0'(\mathbb{R})$, are not stable. Hence, there is no f that identifies $\mathcal{H}_U$ and identification of $\mathcal{H}_M$ is impossible (see Figure 5).

The synthesis map E is given by (19) and is defined to be a linear combination of operators in $\mathcal{H}_U$ whose spreading functions have the form

$$M_{(\lambda K k, \frac{\lambda L}{K} l)}\, T_{(\frac{1}{K} m, \frac{K}{L} n)}\, \eta_P$$

for some well-chosen $\lambda > 1$ (see Lemma 4.5), $k, l, m, n \in \mathbb{Z}$ and P a time-frequency localization operator in $\mathcal{H}_{R_{K,L}}$. Hence the boundedness and stability properties of E can be deduced from the corresponding properties of Gabor systems on $L^2(\mathbb{R}^2)$ with compactly supported window functions (see Section II-A).

The analysis map C_g given by (20) is a standard analysis map for a Gabor system of the form $\{M_{ka'} T_{lb'} g_0\}_{k,l \in \mathbb{Z}}$ where $a' = \lambda^2 K$ and $b' = \frac{\lambda^2 L}{KJ}$, and $g_0(x) = e^{-\pi x^2}$.

The resulting operator $C_g \circ \Phi_f \circ E$ can be represented in the standard basis for $l_0(\mathbb{Z}^2)$ and $l^2(\mathbb{Z}^2)$ as a bi–infinite matrix which is dominated by a skew diagonal. Such matrices are shown in Lemma 4.6 to be unstable, that is, to not possess a continuous inverse.

B. Proof of Theorem 4.1

The proof will require the following basic results.

Definition 4.2: A sequence $\{f_n\}$ in a Hilbert space H is a *Riesz basis for its closed linear span* if there exist constants $c_1, c_2 > 0$ such that for every finite sequence $\{c_n\}$

$$c_1 \sum_n |c_n|^2 \le \Big\| \sum_n c_n\, f_n \Big\|_H^2 \le c_2 \sum_n |c_n|^2.$$

Theorem 4.3: ([35]–[37]) Given $a', b' > 0$ with $a'b' < 1$ and $g_0(x) = e^{-\pi x^2}$, the Gabor system $\{M_{ka'} T_{lb'} g_0\}_{k,l \in \mathbb{Z}}$ is a frame for $L^2(\mathbb{R})$ (see Section II-A).

We begin with a result concerning the composition of Hilbert–Schmidt operators with time–frequency shifts.

Lemma 4.4: Let $P \in \mathcal{H}$ with spreading function $\eta_P \in S_0(\mathbb{R} \times \widehat{\mathbb{R}})$. For $p, r \in \mathbb{R}$ and $\omega, \xi \in \widehat{\mathbb{R}}$, define $\widetilde{P} = M_\omega T_{p-r} P T_r M_{\xi-\omega} \in HS$. Then $\eta_{\widetilde{P}} = e^{2\pi i r\xi} M_{(\omega,r)}\, T_{(p,\xi)}\, \eta_P$ and $\widetilde{P} \in \mathcal{H}$.

Proof: Note that for any $f, g \in S_0(\mathbb{R})$ and $P \in \mathcal{H}$ we have by (5) that

$$\langle Pf, g \rangle = \langle \eta_P, V_f g \rangle$$

where $V_f g(t,\nu) = \langle g, T_t M_\nu f \rangle$, $t \in \mathbb{R}$ and $\nu \in \widehat{\mathbb{R}}$. The interchange of order of integration is justified since f, g, η are in the Feichtinger algebra.

Hence, for $f, g \in S_0(\mathbb{R})$ and $s, r \in \mathbb{R}$ and $\omega, \rho \in \widehat{\mathbb{R}}$ we have

$$\begin{aligned} \langle M_\omega T_s P T_r M_\rho f, g \rangle &= \langle P T_r M_\rho f, T_{-s} M_{-\omega} g \rangle \\ &= \langle \eta_P, V_{T_r M_\rho f} T_{-s} M_{-\omega} g \rangle, \end{aligned}$$

and

$$\begin{aligned} & V_{T_r M_\rho f} T_{-s} M_{-\omega} g(t,\nu) \\ &= \langle T_{-s} M_{-\omega} g, T_t M_\nu T_r M_\rho f \rangle = \langle g, M_\omega T_s T_t M_\nu T_r M_\rho f \rangle \\ &= e^{-2\pi i \omega(s+t)} \langle g, T_{s+t} M_{\omega+\nu} T_r M_\rho f \rangle \\ &= e^{-2\pi i (\omega(s+t) + (\omega+\nu) r)} \langle g, T_{t+r+s} M_{\nu+\rho+\omega} f \rangle \\ &= e^{-2\pi i \omega(s+r)} e^{-2\pi i(\omega t + \nu r)} V_f g(t + (r+s), \nu + (\rho+\omega)). \end{aligned}$$

We have

$$\begin{aligned} & \langle M_\omega T_s P T_r M_\rho f, g \rangle \\ &= \langle \eta_P, e^{-2\pi i \omega(s+r)} M_{-(\omega,r)} T_{-(r+s,\rho+\omega)} V_f g \rangle \\ &= \langle e^{2\pi i \omega(s+r)} T_{(r+s,\omega+\rho)} M_{(\omega,r)} \eta_P, V_f g \rangle \\ &= \langle \eta_R, V_f g \rangle, \end{aligned}$$

where

$$\begin{aligned} \eta_R &= e^{2\pi i \omega(s+r)} T_{(r+s,\omega+\rho)} M_{(\omega,r)} \eta_P \\ &= e^{2\pi i r(\rho+\omega)} M_{(\omega,r)} T_{(r+s,\omega+\rho)} \eta_P. \end{aligned}$$

The choice $s = p - r$ and $\rho = \xi - \omega$ concludes the proof. ∎

Lemma 4.5: Fix $\lambda > 1$ with $1 < \lambda^4 < \frac{J}{L}$ and choose $\eta_1, \eta_2 \in \mathcal{S}(\mathbb{R})$ with values in $[0,1]$ and

$$\eta_1(t) = \begin{cases} 1 & \text{for } t \in [\frac{\lambda-1}{2\lambda K}, \frac{\lambda+1}{2\lambda K}] \\ 0 & \text{for } t \notin [0, \frac{1}{K}] \end{cases}$$

and

$$\eta_2(\nu) = \begin{cases} 1 & \text{for } \nu \in [\frac{(\lambda-1)K}{2\lambda L}, \frac{(\lambda+1)K}{2\lambda L}] \\ 0 & \text{for } \nu \notin [0, \frac{K}{L}] \end{cases} .$$

Let $\eta_P = \eta_1 \otimes \eta_2$. Then $\operatorname{supp} \eta_P \subseteq [0, \frac{1}{K}] \times [0, \frac{K}{L}] = R_{K,L}$ and the operator $P \in \mathcal{H}_{R_{K,L}}$ has the following properties.

a) The operator family

$$\Big\{ M_{\lambda K k}\, T_{\frac{1}{K}m - \frac{\lambda L}{K} l}\, P\, T_{\frac{\lambda L}{K} l}\, M_{\frac{K}{L}n - \lambda K k} \Big\}_{k,l,m,n \in \mathbb{Z}}$$

is a Riesz basis for its closed linear span in $HS(\mathbb{R})$.

b) The operator $P \in \mathcal{H}_{R_{K,L}}$ is a time–frequency localization operator in the following sense: There exist functions $d_1, d_2 : \mathbb{R} \to \mathbb{R}_0^+$, which decay rapidly at infinity, that is, $d_1, d_2 = O(x^{-n})$ for all $n \in \mathbb{N}$, and which have the property that for all $f \in S_0'(\mathbb{R})$ we have $|Pf(x)| \le \|f\|_{S_0'}\, d_1(x)$, $x \in \mathbb{R}$ and $|\widehat{Pf}(\xi)| \le \|f\|_{S_0'}\, d_2(\xi)$, $\xi \in \widehat{\mathbb{R}}$.

Proof: a) Lemma 4.4 implies that

$$\Big\{ M_{\lambda K k}\, T_{\frac{1}{K}m - \frac{\lambda L}{K} l}\, P\, T_{\frac{\lambda L}{K} l}\, M_{\frac{K}{L}n - \lambda K k} \Big\}_{k,l,m,n \in \mathbb{Z}}$$

is a Riesz basis for its closed linear span in $HS(\mathbb{R})$ if and only if

$$\Big\{ M_{\left(\lambda K k, \frac{\lambda L}{K} l\right)}\, T_{\left(\frac{1}{K}m, \frac{K}{L}n\right)}\, \eta_P \Big\}_{k,l,m,n \in \mathbb{Z}}$$

is a Riesz basis for its closed linear span in $L^2(\mathbb{R}\times\widehat{\mathbb{R}})$.

We observe that

$$\begin{aligned} &\| \sum_{k,l,m,n\in\mathbb{Z}} \sigma_{k,l,m,n} M_{\left(\lambda K k, \frac{\lambda L}{K} l\right)}\, T_{\left(\frac{1}{K}m, \frac{K}{L}n\right)}\, \eta_P \|^2_{L^2(\mathbb{R}^2)} \\ &= \sum_{m,n\in\mathbb{Z}} \| \sum_{k,l\in\mathbb{Z}} \sigma_{k,l,m,n}\, M_{\left(\lambda K k, \frac{\lambda L}{K} l\right)}\, T_{\left(\frac{1}{K}m, \frac{K}{L}n\right)}\, \eta_P \|^2_{L^2} \\ &= \sum_{m,n\in\mathbb{Z}} \| \sum_{k,l\in\mathbb{Z}} \sigma_{k,l,m,n}\, M_{\left(\lambda K k, \frac{\lambda L}{K} l\right)}\, \eta_P \|^2_{L^2} \end{aligned}$$

where we have used the translation invariance of the L^2 norm and the fact that the support of η_P is contained in $R_{K,L}$. With $R^\lambda_{K,L} = [\frac{\lambda-1}{2\lambda K}, \frac{\lambda+1}{2\lambda K}] \times [\frac{(\lambda-1)K}{2\lambda L}, \frac{(\lambda+1)K}{2\lambda L}]$, we have

$$\begin{aligned} &\sum_{m,n\in\mathbb{Z}} \| \sum_{k,l\in\mathbb{Z}} \sigma_{k,l,m,n}\, M_{\left(\lambda K k, \frac{\lambda L}{K} l\right)}\, \eta_P \|^2_{L^2} \\ &\asymp \sum_{m,n\in\mathbb{Z}} \| \sum_{k,l\in\mathbb{Z}} \sigma_{k,l,m,n}\, M_{\left(\lambda K k, \frac{\lambda L}{K} l\right)}\, \mathbf{1}_{R^\lambda_{K,L}} \|^2_{L^2(\mathbb{R}^2)} \\ &= \tfrac{1}{\lambda^2 L}\, \|\{\sigma_{k,l,m,n}\}\|^2_{l^2} \end{aligned}$$

since by definition η_P is bounded below by $\mathbf{1}_{R^\lambda_{K,L}}$ and bounded above by the characteristic function of finitely many translates of $R^\lambda_{K,L}$.

b) See [7], Lemma 3.4. ∎

Lemma 4.6 generalizes the fact that $m \times n$ matrices with $m < n$ have a non–trivial kernel and, therefore, are not stable as operators acting on $\mathbb{C}^n$. In fact, the bi–infinite matrices $\mathcal{M} = (m_{j',j})_{j',j\in\mathbb{Z}^2}$ considered in Lemma 4.6 are not dominated by their diagonals $m_{j,j}$ — which would correspond to square matrices — but by skew diagonals $m_{j,\lambda j}$, with $\lambda > 1$, that is, $m_{j',j}$ is small if $\|\lambda j' - j\|_\infty$ is large. The lemma is proven in [7]. The validity of Lemma 4.6, and therefore of Theorem 4.1 does not depend on the choice of (reasonable) topologies on domain and range. In fact, a more general version of Lemma 4.6 can be found in [38].

Lemma 4.6: Given $\mathcal{M} = (m_{j',j}) : l^2(\mathbb{Z}^2) \to l^2(\mathbb{Z}^2)$. If there exists a monotonically decreasing function $w : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ with $w = O\left(x^{-2-\delta}\right)$, $\delta > 0$, and constants $\lambda > 1$ and $K_0 > 0$ with $|m_{j',j}| < w(\|\lambda j' - j\|_\infty)$ for $\|\lambda j' - j\|_\infty > K_0$, then $\mathcal{M}$ is not stable, that is, for every $\epsilon > 0$ there is a $\sigma \in l^2(\mathbb{Z}^2)$ with $\|\sigma\|_{l^2(\mathbb{Z}^2)} = 1$ such that $\|\mathcal{M}\sigma\|_{l^2(\mathbb{Z}^2)} < \epsilon$.

Now all pieces are in place to prove Theorem 4.1.

Proof of Theorem 4.1. Choose λ, η_1, η_2, P, d_1, and d_2 as in Lemma 4.5.

Define the synthesis operator $E : l_0(\mathbb{Z}^2) \to \mathcal{H}_U$ as follows. For $\sigma = \{\sigma_{k,p}\} \in l^2(\mathbb{Z}^2)$ write $\sigma_{k,p} = \sigma_{k,lJ+j}$ for $l \in \mathbb{Z}$ and $0 \le j < J$ and define

$$E(\sigma) = \sum_{k,l\in Z} \sum_{j=0}^{J-1} \sigma_{k,lJ+j} M_{\lambda K k}\, T_{\frac{1}{K}k_j + \frac{\lambda L}{K} l}\, P\, T_{-\frac{\lambda L}{K} l}\, M_{\frac{K}{L}p_j - \lambda K k} \quad (19)$$

Since

$$\Big\{ M_{\lambda K k}\, T_{\frac{1}{K}m - \frac{\lambda L}{K} l}\, P\, T_{\frac{\lambda L}{K} l}\, M_{\frac{K}{L}n - \lambda K k} \Big\}_{k,l,m,n \in \mathbb{Z}}$$

is a Riesz basis for its closed linear span in $HS(\mathbb{R})$, the subset

$$\Big\{ M_{\lambda K k}\, T_{\frac{1}{K}k_j + \frac{\lambda L}{K} l}\, P\, T_{-\frac{\lambda L}{K} l}\, M_{\frac{K}{L}p_j - \lambda K k} \Big\}_{k,l \in \mathbb{Z}, 0\le j<J}$$

is a Riesz basis for its closed linear span in $\mathcal{H}_U \subseteq HS(\mathbb{R})$. We conclude that E is bounded and stable.

To construct a stable (g, a', b')–analysis operator C_g, we choose as Gabor atom the Gaussian $g_0 : \mathbb{R} \to \mathbb{R}^+$, $x \mapsto e^{-\pi x^2}$. By Theorem 4.3, the Gabor system $(g_0, a', b') = \{M_{ka'} T_{lb'} g_0\}$ is a frame for any $a', b' > 0$ with $a'b' < 1$, and we conclude that the analysis map given by

$$\begin{aligned} C_{g_0} :\ \ & L^2(\mathbb{R}) \ \to l^2(\mathbb{Z}^2), \\ & f \ \ \mapsto \Big\{ \langle f, M_{\lambda^2 K k} T_{\frac{\lambda^2 L}{KJ} l}\, g_0 \rangle \Big\}_{k,l} \end{aligned} \quad (20)$$

is bounded and stable since $\lambda^2 K \frac{\lambda^2 L}{KJ} = \lambda^4 \frac{L}{J} < 1$.

For simplicity of notation, set $\alpha = K$ and $\beta = \frac{L}{KJ}$. Fix $f \in S_0'(\mathbb{R})$ and consider the composition

$$\begin{array}{ccccccc} l_0(\mathbb{Z}^2) & \xrightarrow{E} & \mathcal{H}_M & \xrightarrow{\Phi_f} & L^2 & \xrightarrow{C_{g_0}} & l^2(\mathbb{Z}^2) \\ \sigma & \mapsto & E\sigma & \mapsto & E\sigma\, f & \mapsto & \\ & & & & \multicolumn{3}{l}{\big\{ \langle E\sigma\, f,\ M_{\lambda^2 \alpha k'} T_{\lambda^2 \beta l'}\, g_0 \rangle \big\}_{k',l'} .} \end{array}$$

Since

$$\begin{aligned}
&\Big(C_{g_0}\circ\Phi_f\circ E\;\{\sigma_{k,lJ+j}\}\Big)_{k',l'}\\
&= \Big\langle \sum_{k,l}\sum_{j=0}^{J-1} \sigma_{k,lJ+j}\; M_{\lambda\alpha k}\, T_{\frac{k_j}{\alpha}+\lambda\beta lJ}\, P\, T_{-\lambda\beta lJ}\, M_{\frac{p_j}{\beta J}-\lambda\alpha k}\, f\,,\; M_{\lambda^2\alpha k'}T_{\lambda^2\beta l'}g_0\Big\rangle\\
&= \sum_{k,l}\sum_{j=0}^{J-1}\Big\langle M_{\lambda\alpha k}\, T_{\frac{k_j}{\alpha}+\lambda\beta lJ}\, P\, T_{-\lambda\beta lJ}\, M_{\frac{p_j}{\beta J}-\lambda\alpha k}\, f\,,\; M_{\lambda^2\alpha k'}T_{\lambda^2\beta l'}g_0\Big\rangle\sigma_{k,lJ+j}\\
&= \sum_{k,l}\sum_{j=0}^{J-1} m_{k',l',k,lJ+j}\;\sigma_{k,lJ+j}\,,
\end{aligned}$$

we see that the operator $C_{g_0}\circ\Phi_f\circ E$ is represented — with respect to the canonical basis $\{\delta(\cdot-n)\}_n$ of $l^2(\mathbb{Z}^2)$ — by the bi–infinite matrix

$$\begin{aligned}
\mathcal{M} &= \Big(m_{k',l',\,k\,,lJ+j}\Big)\\
&= \Big(\Big\langle\; M_{\lambda\alpha k}\, T_{\frac{k_j}{\alpha}+\lambda\beta lJ}\, P\, T_{-\lambda\beta lJ}\, M_{\frac{p_j}{\beta J}-\lambda\alpha k}\, f\,,\; M_{\lambda^2\alpha k'}T_{\lambda^2\beta l'}\,g_0\Big\rangle\Big)\,.
\end{aligned}$$

We shall now use Lemma 4.6 to show that $\mathcal{M}$, and, therefore, $C_{g_0}\circ\Phi_f\circ E$ is not stable. Lemma 4.5, part *b*, together with the rapidly decaying function

$$\widetilde{d_1} = \max_{j=0,\dots,J-1} T_{\frac{k_j}{\alpha}-\lambda\beta j}d_1$$

will provide us with the necessary bounds on the matrix entries of $\mathcal{M}$. In fact, for $k,l,k',l'\in\mathbb{Z}$ and $0\le j<J$, we have

$$\begin{aligned}
&|m_{k',l',k,lJ+j}|\\
&= \Big|\Big\langle M_{\lambda\alpha k}\, T_{\frac{k_j}{\alpha}+\lambda\beta lJ}\, P\, T_{-\lambda\beta lJ}\, M_{\frac{p_j}{\beta J}-\lambda\alpha k}\, f\,,\; M_{\lambda^2\alpha k'}T_{\lambda^2\beta l'}g_0\Big\rangle\Big|\\
&\le \Big\langle T_{\lambda\beta(lJ+j)}\big(T_{\frac{k_j}{\alpha}-\lambda\beta j}\Big|P\, T_{-\lambda\beta lJ}\, M_{\frac{p_j}{\beta J}-\lambda\alpha k}\, f\Big|\big)\,,\; T_{\lambda^2\beta l'}g_0\Big\rangle\\
&\le \|f\|_{S_0'}\,\Big\langle T_{\lambda\beta(lJ+j)}T_{\frac{k_j}{\alpha}-\lambda\beta j}d_1\,,\; T_{\lambda^2\beta l'}g_0\Big\rangle\\
&\le \|f\|_{S_0'}\,\Big\langle T_{\lambda\beta(lJ+j)}\widetilde{d_1}\,,\; T_{\lambda^2\beta l'}g_0\Big\rangle\\
&\le \|f\|_{S_0'}\,(\widetilde{d_1}*g_0)\,(\lambda\beta(\lambda l'-(lJ+j)))\,,
\end{aligned}$$

and

$$\begin{aligned}
&|m_{k',l',k,lJ+j}|\\
&= \Big|\Big\langle T_{\lambda\alpha k}\, M_{-\frac{k_j}{\alpha}-\lambda\beta lJ}\,\big(P\, T_{-\lambda\beta lJ}\, M_{\frac{p_j}{\beta J}-\lambda\alpha k}\, f\big)\widehat{\ },\; T_{\lambda^2\alpha k'}M_{-\lambda^2\beta l'}g_0\Big\rangle\Big|\\
&\le \Big\langle T_{\lambda\alpha k}\Big|\big(P\, T_{-\lambda\beta lJ}\, M_{\frac{p_j}{\beta J}-\lambda\alpha k}\, f\big)\widehat{\ }\Big|\,,\; T_{\lambda^2\alpha k'}g_0\Big\rangle\\
&\le \|f\|_{S_0'}\,(d_2*g_0)(\lambda\alpha(\lambda k'-k)).
\end{aligned}$$

where we have used the Parseval–Plancherel identity and the fact that $g_0\ge 0$, $\widehat{g_0}=g_0$, and $g_0(-x)=g_0(x)$. Since $\widetilde{d_1}$, d_2, and g_0 decay rapidly, so do $\widetilde{d_1}*g_0$ and d_2*g_0, that is, $d_1*g_0, d_2*g_0 = O(x^{-n})$ for all $n\in\mathbb{N}$. We set

$$\begin{aligned}
w(x) = \|f\|_{S_0'}\;\max\big\{&\widetilde{d_1}*g_0(\lambda\beta x)\,,\;\widetilde{d_1}*g_0(-\lambda\beta x)\,,\\
&d_2*g_0(\lambda\alpha x),\; d_2*g_0(-\lambda\alpha x)\big\}.
\end{aligned}$$

and obtain $|m_{k',l',k,l}|\le w\big(\max\{|\lambda k'-k|,\;|\lambda l'-l|\}\big)$ with $w=O(x^{-n})$ for $n\in\mathbb{N}$. Lemma 4.6 implies that $\mathcal{M}$ is not stable, and, by construction, we can conclude that $C_{g_0}\circ\Phi_f\circ E$ and thus Φ_f is not stable. ■

V. Conclusions

In this paper we have provided a proof of a conjecture made by Bello in [8] on the relationship between the identifiability of a time varying communication channel and the size of the support of the spreading function of the channel. Bello's conjecture is a generalization of a similar conjecture made by Kailath in [3]. Kailath's conjecture was proved recently in [7].

The conjecture states roughly that a communication channel modelled by the operator H and with spreading function η_H is identifiable if $\mathrm{vol}\,(\mathrm{supp}\,\eta_H)<1$ and is not identifiable if $\mathrm{vol}\,(\mathrm{supp}\,\eta_H)>1$. It is not known what happens when the volume of the spreading support is exactly 1.

The proof of sufficiency, that is, the proof that a channel is identifiable if the volume of its spreading support is less than 1 is constructive in the sense that it suggests a way to construct the spreading function for the channel operator H from the measurement of Hf where the identifier f is a delta train weighted by a periodic sequence. While the paper does not give a closed form formula for the spreading function, it should be possible to design an algorithm to recover the spreading function from the channel measurement.

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