TIME-FREQUENCY ANALYSIS: TUTORIAL

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Overview

- TF-Analysis: Spectral Visualization of nonstationary signals (speech, audio, ...)
  - Spectrogram (time-varying spectrum estimation)

- TF-methods for signal processing:
  - Ambiguity function (range/Doppler estimation)
  - Short-time Fourier transform (LTV filter design)

- TF-representation of underspread linear operators:
  - Spreading Function (representation & classification)
  - Kohn-Nirenberg symbol (LTV transfer function)
  - Application: MIMO-based SAR radar problem
Mathematical Setup

• Classical Theory:
  – signals defined on the real line
  – Hilbert space setup usual (Math. Physics and EE)
  – Gelfand brackets (pure mathematics)

• Numerical Practice:
  – signals are vectors in $\mathbb{C}^N$
  – Fourier Transform $= \text{DFT} = \text{realized by FFT}$

• Open Problems:
  – Algebraic & Number theoretic methods
  – try to take finite alphabet effects in account
Time-Frequency Shift

- Unitary time-frequency shift operator

\[(U(\tau, \nu)x)(t) = x(t - \tau) \exp(2\pi i \nu t)\]

- Superposition Law (Schrödinger Repr. of WH-Group)

\[(U(\tau_1, \nu_1)U(\tau_2, \nu_2)x)(t) = x(t - (\tau_1 + \tau_2)) \exp(2\pi i ((n_1 + n_2)t - \nu_2\tau_1))\]

- NO unitary group representation of \(\mathbb{R} \times \mathbb{R}\)
Short-Time Fourier Transform

- Sliding a window \( g(t) \) along the signal followed by Fourier transform of the windowed partial signal

\[
(V_g x)(t,f) = \int x(t') \overline{g(t-t')} \exp(-2\pi i ft')
\]
Spectrogram

• The Short-time Fourier transform is complex valued and its real part and imaginary part are highly oscillatory

• adequate visualization is given by the squared magnitude => Spectrogram

•

\[(S_g x)(t,f) = |(V_g x)(t,f)|^2\]

• The spectrogram can be interpreted as a smoothed Wigner distribution
Spectrogram: „Short“ Window
Spectrogram: „Long“ Window
STFT-based Filtering

- Reconstruction of signal from STFT:

\[ x(t) = \int \int V_g(t', f')(U(t', f')g)(t) \, dt' \, df' \]

- Reconstruction of signal from multiplicatively modified STFT:

\[ (Hx)(t) = \int \int M(t', f')V_g(t', f')(U(t', f')g)(t) \, dt' \, df' \]

- this allows synthesis of HS operator (LTV filter) based on the time-frequency model \( M(t, f) \)
Radar Ambiguity Function

- How behaves the inner product of a signal and its TF-shifted version \( \Rightarrow \) time-frequency correlation function
- Well-known as Radar ambiguity function

\[
(A_x)(\tau, \nu) = \int x(t) \overline{x(t-\tau)} \exp(-2\pi i \nu t) \, dt
\]

- Radar uncertainty principle:

\[
\int \int |(A_x)(\tau, \nu)|^2 \, d\tau \, d\nu = \|x\|^4
\]

\[
|(A_x)(0,0)|^2 = \|x\|^4
\]
Range-Doppler Radar

Transmitted signal

\[ U(\tau, \nu)x \] ..... reflected signal

Range

Doppler
Range-Doppler Estimation

- The peak of the cross-ambiguity function is a ML-estimate for the Range-Doppler

\[(\tau, \nu)_{est} = \text{argmax} \left( A_{y,x}(\tau, \nu) \right)\]

- Curvature of Ambiguity function of x determines the Cramer-Rao bound for range-Doppler estimation => we want a peaky signal, however one has:

\[
\frac{\partial^2 A_x}{\partial \nu^2}(0,0) = -4 \pi^2 \int t^2 |x(t)|^2 dt
\]

\[
\frac{\partial^2 A_x}{\partial \tau^2}(0,0) = -4 \pi^2 \int f^2 |X(f)|^2 df
\]
Radar Synthesis Problem

- Ambiguity function is quadratic signal representation $\Rightarrow$ inner symmetry, i.e. an arbitrary function is no valid ambiguity function

- Given a nonvalid time-frequency model how can we determine the closest valid ambiguity function

$$x_{opt} = \arg \min_x \| A_x - M \|^2$$

- Boils down to a partial eigenvalue problem of a self-adjoint matrix:

$$Q(M)x_{opt} = \lambda_{\text{max}} x_{opt}$$
Spreading Function

- Decomposition of linear operator into a superposition of time-frequency shift operators

\[(S_H)(\tau, \nu) = \int H(t, t - \tau) \exp(-2\pi i \nu t) \, dt\]

- Inner product representation

\[(S_H)(\tau, \nu) = \langle H, U(\tau, \nu) \rangle\]
Kohn-Nirenberg Symbol

- Decomposition of linear operator into a superposition of time-frequency shift operators

\[(K_H)(t, f) = \int H(t, t - \tau) \exp(-2\pi if\tau) d\tau\]
Underspread Operators

\[ \tau_{\text{max}} \cdot \nu_{\text{max}} \leq 1 \]

Diagram: A shaded rectangle with vertices at \( (0, \nu_{\text{max}}), (\tau_{\text{max}}, \nu_{\text{max}}), (\tau_{\text{max}}, 0), (0, 0) \), with axes labeled \( \nu \) and \( \tau \).
Underspread Operators
Underspread Asymptotics

- Underspread operators are approximately normal
- Underspread operators do approximately commute
- Underspread operators are approximately diagonalized by a properly adapted Gabor basis
- Underspread operators can be realized as STFT multipliers
Spectrogram: Adapted Window
SAR Radar

- Determine/Classify the whole object rather than its range and velocity from observation of reflected signal
- System identification problem: given $x$ and $Hx$ estimate $H$ and then classify the object based on this estimate
- SAR = Synthetic Aperture Radar
Gabor/STFT based Source Coding