Some Math of the PAR Problem

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1. Introduction

   A pilot tone design problem and PAR optimization

2. Unimodular case

   Littlewood-Erdoes conjectures, ultraflat polynomials

3. $L_p - L_q$ estimates

   “Optimal” PAR not possible for general coefficient moduli but

   PAR penalty only logarithmic in number of frequencies

Problem

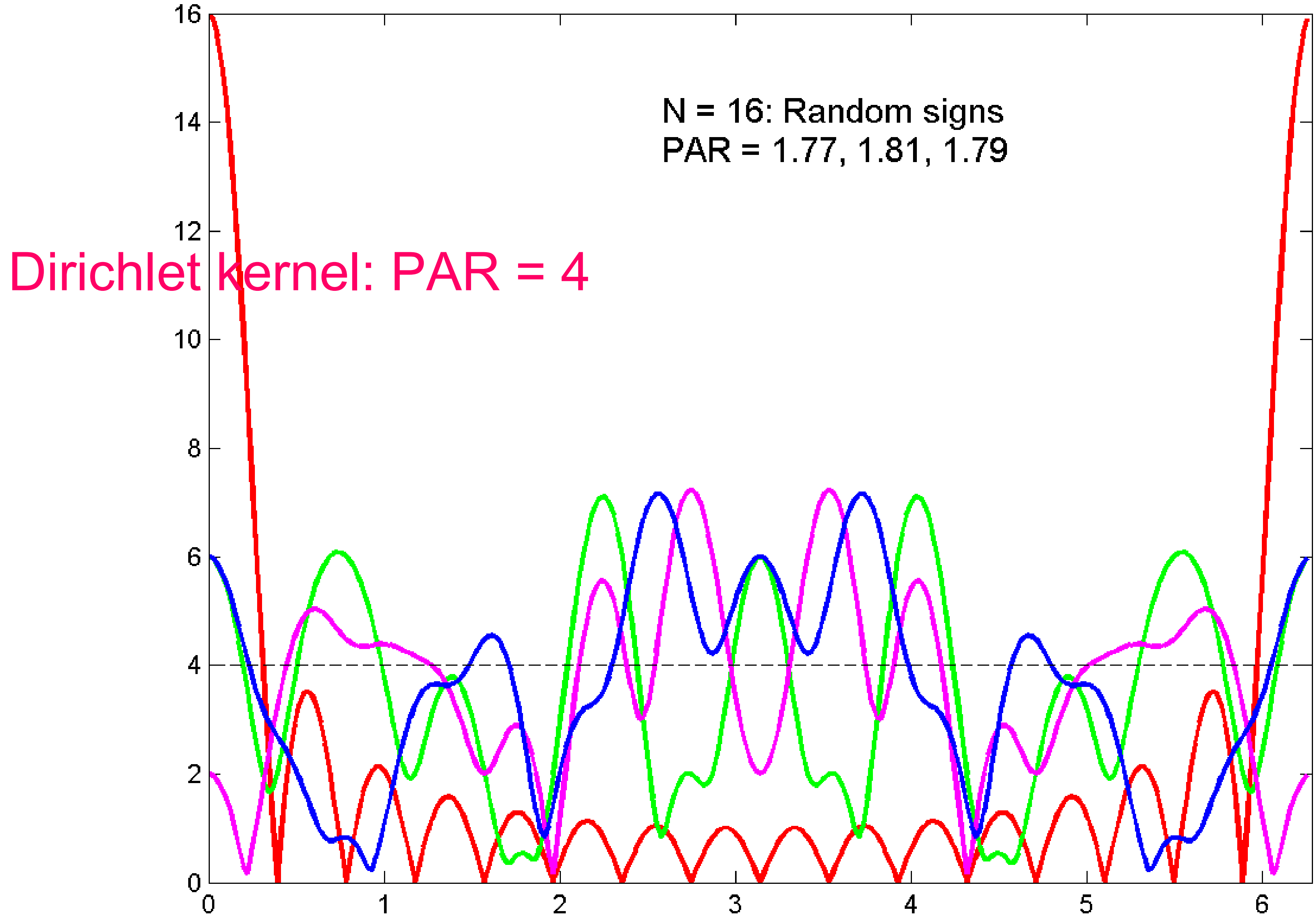
Frequency domain		Time domain
$\{c_k\}$	$\rightarrow$	$P(t) = \sum_{k=0}^{N-1} c_k \exp(ikt)$
$\downarrow$		$\downarrow$
$\left(\sum \lvert c_k \rvert^2\right)^{1/2}$	$\rightarrow$	$\max \lvert P(t) \rvert$
Average power		Peak power (relevant for RF components)
L_2 norm of P(t)		L_∞ norm of P(t)

Note: For math convenience, we use sqrt of what power means to the engineer!

$$\boxed{\text{PAR} = \frac{\text{Peak power}}{\text{Average power}} \longrightarrow \min}$$

at least reduced !

Example: $c_k = 1$



Dirichlet kernel: PAR = 4
N = 16: Random signs
PAR = 1.77, 1.81, 1.79
16
14
12
10
8
6
4
2
0
0
1
2
3
4
5
6

Pilot tones for time synchronization

Suppose, we send a specific signal (known to the receiver)

$$P(t) = \sum_{k=0}^{N-1} c_k \exp(ikt)$$

and it arrives time-delayed (by an unknown delay τ).
To synchronize at the receiver, we perform convolution

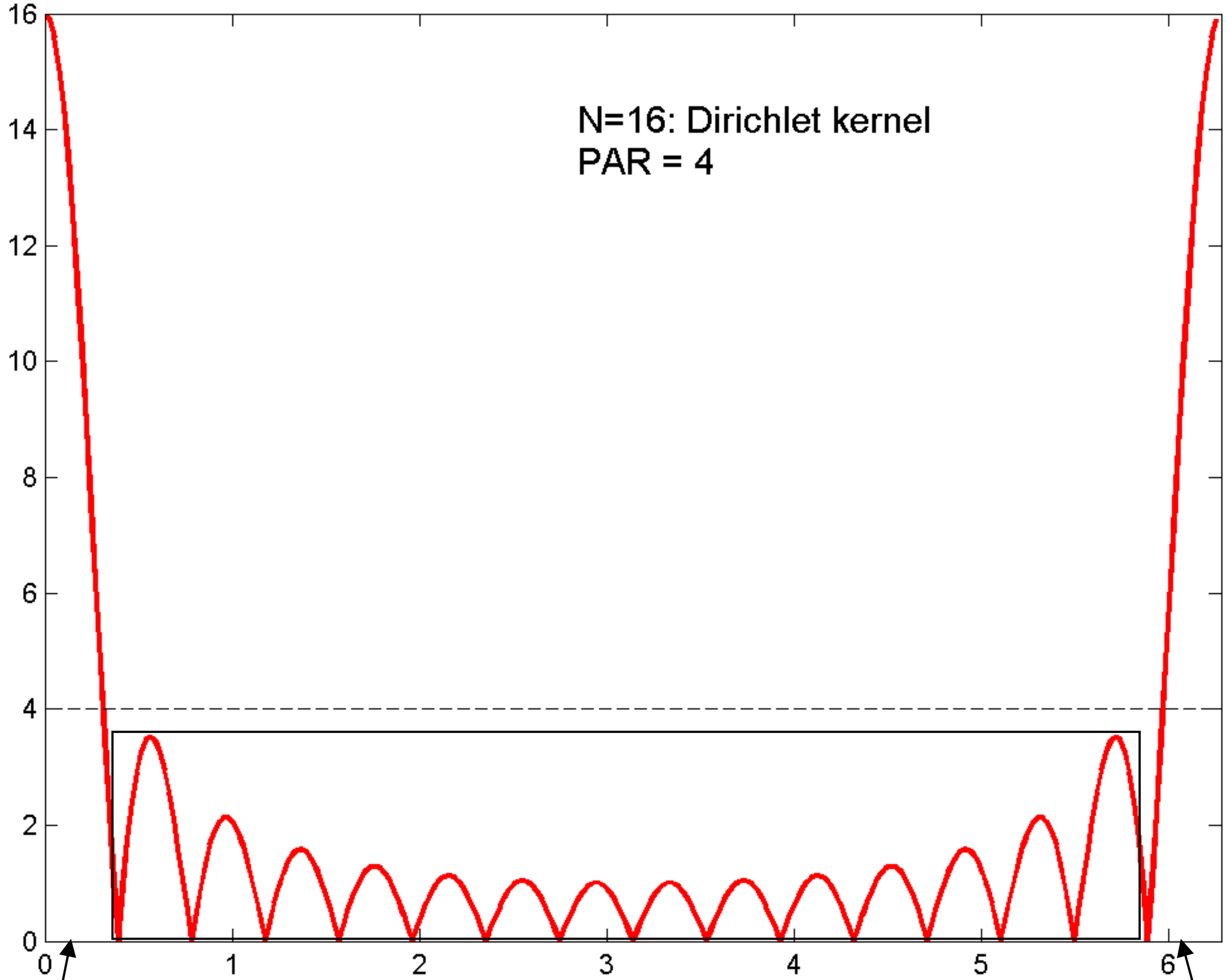
$$\int_0^{2\pi} P(t-\tau) \cdot \overline{P(t-s)}\, ds = \sum_{k=0}^{N-1} |c_k|^2 \exp(ik(s-\tau))$$

and can use this autocorrelation function to detect τ if it is peaky. E.g., the Dirichlet kernel

$$P(t) = \sum_{k=0}^{N-1} \exp(ikt), \quad r_k := |c_k|^2 = 1$$

or the so-called Dolph-Cheyshev design can be used.

Dirichlet kernel ($r_k = 1$)



$d=2\pi/N$

d

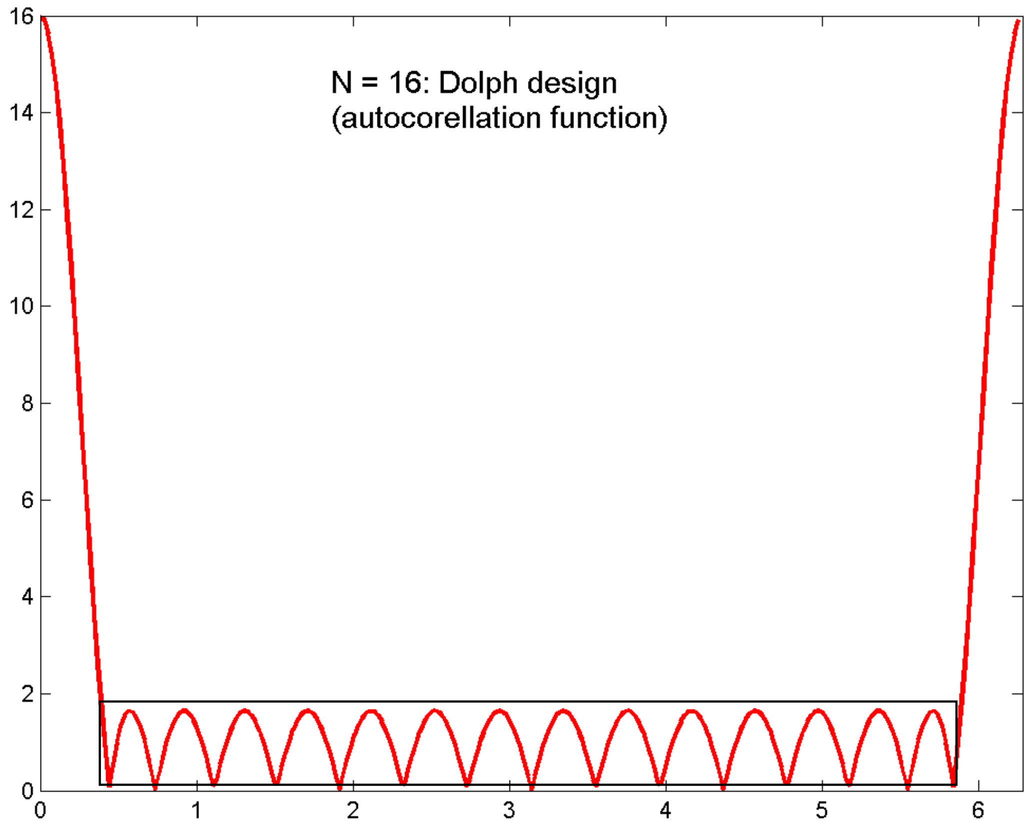
For this d choice:

$$\frac{\text{Side lobe maximum}}{\text{Peak value } P(0)} \longrightarrow 0.2122\ldots$$

as N goes to infty.

Can one do better?

Yes: Dolph-Chebyshev design



For same d choice:

$$\frac{\text{Side lobe maximum}}{\text{Peak value } P(0)} \longrightarrow 1/\cosh(\pi) = 0.0863\ldots$$

Optimal coefficients come from Chebyshev polynomials.!

One problem remains:

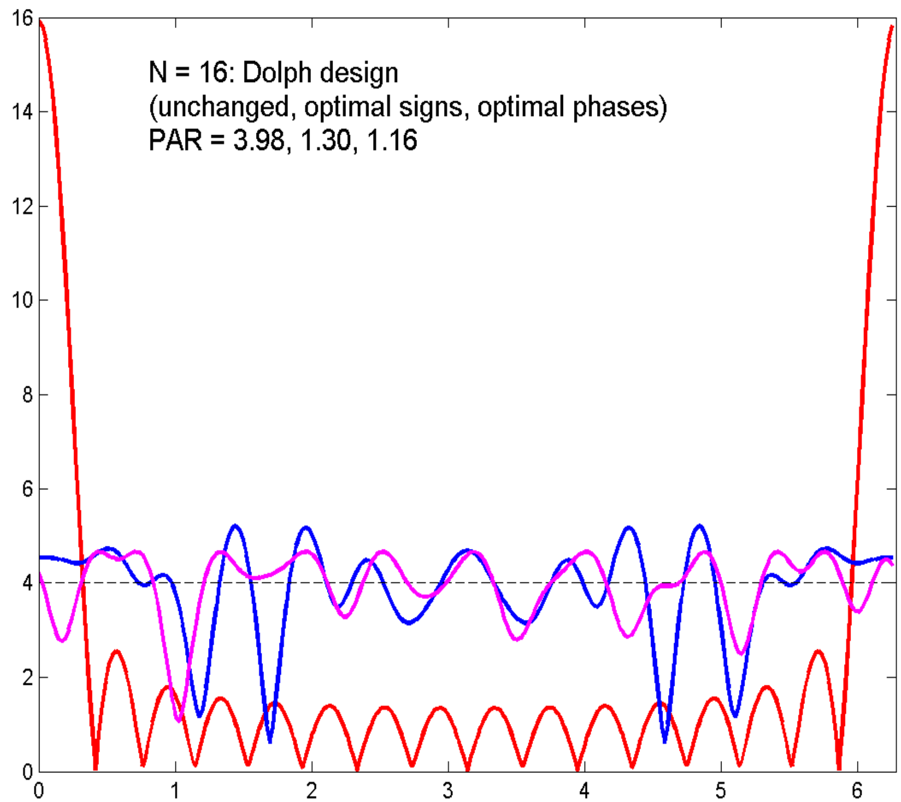
All peaky designs have close to worst PAR value

= cost a lot of powering of RF components!

Same procedure for arbitrary r_k !

All pilot tones of the form $$Q(t) = \sum_{k=0}^{N-1} \sqrt{r_k}\, e^{i\alpha_k} \cdot e^{ikt}$$

have the same autocorrelation function, can freely choose real phases α for PAR minimization!



Talk by Werner Henkel

Be aware:
Real-life PAR reduction is not as easy, too many constraints on what can be changed in a design.

Littlewood-Erdoes conjectures

(harmonic analysis, number theory, probability, optimization,…)

Set $$L_N := \{P(t) = \sum_{k=0}^{N-1} c_k \exp(ikt) \quad : \quad c_k = \pm 1\}$$

$$K_N := \{P(t) = \sum_{k=0}^{N-1} c_k \exp(ikt) \quad : \quad |c_k| = 1\}$$

Flatness conjecture (Littlewood 195x): For some a,A and all large N

$$\exists P(t) \in L_N :\ 0 < a \le \max |P(t)| / \sqrt{N} \le A < \infty$$

Maybe, a,A $\longrightarrow$ 1 as N goes to infinity (ultraflatness)?

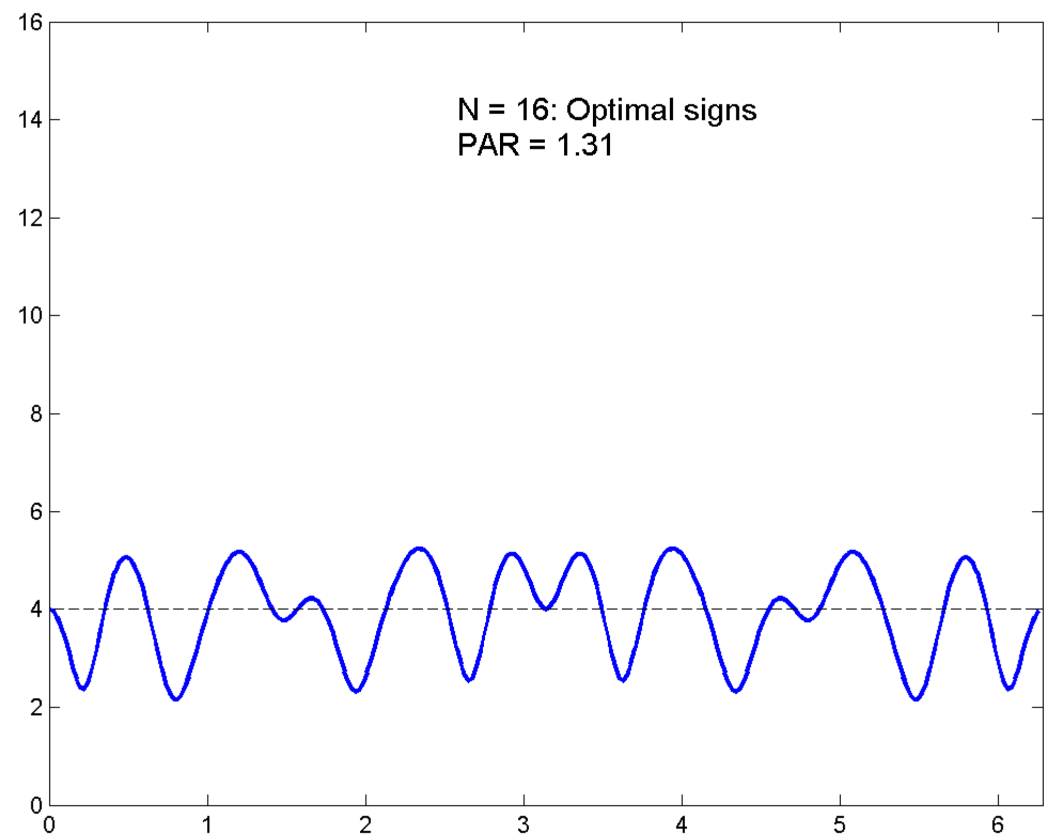
PAR conjecture (Erdoes 1957, 1995: 100$ reward)

$$P(t) \in L_N :\ \max |P(t)| / \sqrt{N} \ge A > 1$$

Known results

- Salem/Zygmund (1954): Average PAR is higher by a logarithmic factor.
- Rudin/Shapiro (~1958): Upper PAR bound $A < 5$. Deterministic construction.
- Littlewood (1961): Asymptotically, $A < 1.36$ using Gauss sums.
- Beller (1971): Similarly, $A < 1.172$.
- Kahane (1980): Ultraflat polynomials exist if arbitrary phases are allowed! Probabilistic arguments.
- Beck (1991): Flat polynomials exist for phases from discrete set $\{2\pi m/400\}$. Methods less probabilistic, don't work for 400 replaced by 2 !
- Odlyzko (199x): Extensive experiments, generally supportive of the flatness conjecture.
- Recent work by Borwein, Erdelyi, Saffari, and others (e.g. Approximation Theory X, 2001, pp. 1-40)

PAR optimal polynomial from L_{16}



The L_p - l_q PAR problem

Denote

$$\mathrm{par}_{p,q} := \max_{0\neq r\in R_+^N} \min_{c\in C^N:|c_k|=r_k} \| \sum_{k=0}^{N-1} c_k e^{ikt} \|_{L_p} / \| r \|_{l_q}$$

This is the worst-case PAR problem (for arbitrarily given coefficient moduli) in more general norms.

Important special case:

$$\mathrm{par}_{\infty,2} \approx \sqrt{\log(N)}, \quad N \to \infty$$

Worst case achieved for lacunary sequences.

Facit: PAR problem is always decently solvable if all phases are at the disposal.

What is known?