Zero localization: from 17th-century algebra to challenges of today

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Bremen summer school

July 2013

Descartes' rule of signs

René Descartes

Theorem [Descartes].

The number of positive zeros of a real univariate polynomial does not exceed the number of sign changes in its sequence of coefficients. Moreover, it has the same parity.

Theorem [Sturm].

The number of zeros of a real univariate polynomial p on the interval $(a, b]$ is given by $V(a) - V(b)$, with $V()$ the number of sign changes in its Sturm sequence $p, p_1, p_2, \ldots$.

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Euclidean algorithm and continued fractions

Starting from $f_0 := p$, $f_1 := q - (b_0/a_0)p$, form the Euclidean algorithm sequence

$$f_{j-1} = q_j f_j + f_{j+1}, \quad j = 1, \ldots, k, \quad f_{k+1} = 0.$$

Then f_k is the greatest common divisor of p and q. This gives a continued fraction representation

$$R(z) = \frac{f_1(z)}{f_0(z)} = \cfrac{1}{q_1(z) + \cfrac{1}{q_2(z) + \cfrac{1}{q_3(z) + \cfrac{1}{\ddots + \cfrac{1}{q_k(z)}}}}}.$$

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Generalized Jacobi matrices

$$\mathcal{J}(z) := \begin{bmatrix} q_k(z) & -1 & 0 & \dots & 0 & 0 \\ 1 & q_{k-1}(z) & -1 & \dots & 0 & 0 \\ 0 & 1 & q_{k-2}(z) & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & q_2(z) & -1 \\ 0 & 0 & 0 & \dots & 1 & q_1(z) \end{bmatrix}.$$

Remark 1. $h_j(z) := f_j(z)/f_k(z)$ is the leading principal minor of $\mathcal{J}(z)$ of order $k-j$. In particular, $h_0(z) = \det \mathcal{J}(z)$.

Remark 2. Eigenvalues of the generalized eigenvalue problem

$$\mathcal{J}(z)u = 0$$

are closely related to properties of $R(z)$.

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Jacobi continued fractions

In the regular case,

$$q_j(z) = \alpha_j z + \beta_j, \qquad \alpha_j, \beta_j \in \mathbb{C}, \alpha_j \neq 0.$$

The polynomials f_j satisfy the three-term recurrence relation
$f_{j-1}(z) = (\alpha_j z + \beta_j) f_j(z) + f_{j+1}(z), \quad j = 1, \ldots, r.$

$$R(z) = \frac{f_1(z)}{f_0(z)} = \cfrac{1}{\alpha_1 z + \beta_1 + \cfrac{1}{\alpha_2 z + \beta_2 + \cfrac{1}{\alpha_3 z + \beta_3 + \cfrac{1}{\ddots + \cfrac{1}{\alpha_r z + \beta_r}}}}}.$$

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In the doubly regular case,

$$\begin{aligned} q_{2j}(z) &= c_{2j}, \qquad j = 1, \ldots \left\lfloor \frac{k}{2} \right\rfloor, \\ q_{2j-1}(z) &= c_{2j-1}z, \qquad j = 1, \ldots, r. \end{aligned}$$

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$$T := \begin{cases} c_{2r} & \text{if } |R(0)| < \infty, \\ c_{2r-1}z & \text{if } R(0) = \infty. \end{cases}$$

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Sturm's algorithm is a variation of the Euclidean algorithm

$$f_{j-1}(z) = q_j(z) f_j(z) - f_{j+1}(z), \quad j = 0, 1, \ldots, k,$$

where $f_{k+1}(z) = 0$. The polynomial f_k is the greatest common divisor of p and q.
The Sturm algorithm is regular if the polynomials q_j are linear.

Theorem [Sturm].

$\mathrm{Ind}_{-\infty}^{+\infty}\left(\frac{f_1}{f_0}\right) = n - 2V(h_0, \ldots, h_n)$ where h_k is the leading coefficient of f_k.

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Cauchy indices

Definition.

$$\mathrm{Ind}_\omega(F) := \begin{cases} +1, & \text{if} \quad F(\omega - 0) < 0 < F(\omega + 0), \\ -1, & \text{if} \quad F(\omega - 0) > 0 > F(\omega + 0), \end{cases}$$

is the index of the function F at its real pole ω of odd order.

Theorem [Gantmacher].

If a rational function R with exactly r poles is represented by a series

$$R(z) = s_{-1} + \frac{s_0}{z} + \frac{s_1}{z^2} + \cdots, \quad \text{then}$$

$$\mathrm{Ind}_{-\infty}^{+\infty} = r - 2V(D_0(R), D_1(R), D_2(R), \ldots, D_r(R)).$$

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Hankel and Hurwitz matrices

Let $R(z)$ be a rational function expanded in its Laurent series at ∞

$$R(z) = s_{-1} + \frac{s_0}{z} + \frac{s_1}{z^2} + \frac{s_2}{z^3} + \cdots .$$

Introduce the infinite Hankel matrix $S := [s_{i+j}]_{i,j=0}^{\infty}$ and consider the leading principal minors of S:

$$D_j(S) := \det \begin{bmatrix} s_0 & s_1 & s_2 & \dots & s_{j-1} \\ s_1 & s_2 & s_3 & \dots & s_j \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ s_{j-1} & s_j & s_{j+1} & \dots & s_{2j-2} \end{bmatrix}, \quad j = 1, 2, 3, \dots .$$

These are Hankel minors or Hankel determinants.

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Hurwitz determinants

$$\text{Let } R(z) = \frac{q(z)}{p(z)}, \quad p(z) = a_0 z^n + \cdots + a_n, \quad a_0 \neq 0,$$
$$q(z) = b_0 z^n + \cdots + b_n,$$

For each $j = 1, 2, \ldots$, denote

$$\nabla_{2j}(p,q) := \det \begin{bmatrix} a_0 & a_1 & a_2 & \ldots & a_{j-1} & a_j & \ldots & a_{2j-1} \\ b_0 & b_1 & b_2 & \ldots & b_{j-1} & b_j & \ldots & b_{2j-1} \\ 0 & a_0 & a_1 & \ldots & a_{j-2} & a_{j-1} & \ldots & a_{2j-2} \\ 0 & b_0 & b_1 & \ldots & b_{j-2} & b_{j-1} & \ldots & b_{2j-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & a_0 & a_1 & \ldots & a_j \\ 0 & 0 & 0 & \ldots & b_0 & b_1 & \ldots & b_j \end{bmatrix}.$$

These are the Hurwitz minors or Hurwitz determinants.

Hankel $\leftrightarrow$ Hurwitz

Theorem [Hurwitz].

Let $R(z) = q(z)/p(z)$ with notation as above. Then

$$\nabla_{2j}(p, q) = a_0^{2j} D_j(R), \quad j = 1, 2, \ldots.$$

Corollary.

Let $T(z) = -1/R(z)$ with notation as above. Then

$$D_j(S) = s_{-1}^{2j} D_j(T), \quad j = 1, 2, \ldots.$$

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Even and odd parts of polynomials

Let p be a real polynomial

$$p(z) = a_0 z^n + a_1 z^{n-1} + \ldots + a_n =: p_0(z^2) + z p_1(z^2), \quad a_0 > 0, \; a_i \in \mathbb{R},$$

Let $n = \deg p$ and $m = \left\lfloor \frac{n}{2} \right\rfloor$.

for $n = 2m$

$$\begin{aligned} p_0(u) &= a_0 u^m + a_2 u^{m-1} + \ldots + a_{n-2} u + a_n, \\ p_1(u) &= a_1 u^{m-1} + a_3 u^{m-2} + \ldots + a_{n-3} u + a_{n-1}, \end{aligned}$$

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Associated function and the main theorem of stability

Introduce the function associated with the polynomial $p(z) = p_0(z^2) + zp_1(z^2)$

$$\Phi(u) = \frac{p_1(u)}{p_0(u)}$$

Definition.

A polynomial is Hurwitz stable if all its zeros lie in the open left half-plane.

Main Theorem of Stability

A polynomial $z \mapsto p(z) = p_0(z^2) + zp_1(z^2)$ is Hurwitz stable if and only if

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Hurwitz and Lienard–Chipart Theorems

Hurwitz Theorem

A polynomial p of degree n is Hurwitz stable if and only if

$$\Delta_1(p) > 0,\ \Delta_2(p) > 0, \ldots,\ \Delta_n(p) > 0.$$

Lienard and Chipart Theorem

A polynomial p of degree n is Hurwitz stable if and only if

$$\Delta_{n-1}(p) > 0,\ \Delta_{n-3}(p) > 0,\ \Delta_{n-5}(p) > 0,\ \ldots$$

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Using properties of functions mapping the UHP to LoHP and the main theorem of stability, one can obtain

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where $c_0 \geqslant 0$ and $c_i > 0$, $i = 1, \ldots, 2m$, and $m = \left\lfloor \frac{n}{2} \right\rfloor$.

Further criteria of stability

Theorem.

A polynomial p of degree n is Hurwitz stable if and only if the infinite Hankel matrix $S = ||s_{i+j}||_{i,j=0}^{\infty}$ is a sign-regular matrix of rank m, where $m = \left\lfloor \frac{n}{2} \right\rfloor$.

Theorem.

A polynomial $f = p(z^2) + zq(z^2)$ is stable if and only if its infinite Hurwitz matrix $H(p, q)$ is totally nonnegative.

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Factorization of infinite Hurwitz matrices

Theorem

If $g(z) = g_0 z^l + g_1 z^{l-1} + \ldots + g_l$, then

$$H(p \cdot g, q \cdot g) = H(p, q)\mathcal{T}(g), \qquad \text{where}$$

$$\mathcal{T}(g) := \begin{bmatrix} g_0 & g_1 & g_2 & g_3 & g_4 & \cdots \\ 0 & g_0 & g_1 & g_2 & g_3 & \cdots \\ 0 & 0 & g_0 & g_1 & g_2 & \cdots \\ 0 & 0 & 0 & g_0 & g_1 & \cdots \\ 0 & 0 & 0 & 0 & g_0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

Here we set $g_i = 0$ for all $i > l$.

Another factorization

Theorem

If the Euclidean algorithm for the pair p, q is doubly regular, then $H(p,q)$ factors as

$$H(p,q) = J(c_1)\cdots J(c_k)H(0,1)\mathcal{T}(g),$$

$$J(c) := \begin{bmatrix} c & 1 & 0 & 0 & 0 & \dots \\ 0 & 0 & 1 & 0 & 0 & \dots \\ 0 & 0 & c & 1 & 0 & \dots \\ 0 & 0 & 0 & 0 & 1 & \dots \\ 0 & 0 & 0 & 0 & c & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \quad H(0,1) = \begin{bmatrix} 1 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & \dots \\ 0 & 1 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 1 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

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where $c_0 \geqslant 0$ and $c_i > 0$, $i = 1, \ldots, 2m$, and $m = \left\lfloor \frac{n}{2} \right\rfloor$.

Stable polynomials and Jacobi continued fractions

"Jacobi" criterion of stability

A polynomial p of degree n is Hurwitz stable if and only if its associated function Φ has the following Jacobi continued fraction expansion

$$\Phi(u) = \frac{p_1(u)}{p_0(u)} = -\alpha u + \beta + \cfrac{1}{\alpha_1 u + \beta_1 - \cfrac{1}{\alpha_2 u + \beta_2 - \cfrac{1}{\ddots + \cfrac{1}{\alpha_n u + \beta_n}}}},$$

where $\alpha \geq 0$, $\alpha_j > 0$ and $\beta, \beta_j \in \mathbb{R}$.

Other topics

- Stielties, Jacobi, other continued fractions. Padé approximation
- Orthogonal polynomials. Moment problems
- Nevanlinna functions
- Pólya frequency sequences and functions
- Laguerre-Pólya class and its generalizations
- Total nonnegativity
- Matrix factorizations
- Hurwitz rational and meromorphic functions

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References

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- *"Lectures on the Routh-Hurwitz problem",* Yu. S. Barkovsky, translated by O. H. & M. Tyaglov, arXiv:0802.1805.
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