Zero localization: from 17th-century algebra to challenges of today

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Descartes’ rule of signs

Theorem [Descartes].
The number of positive zeros of a real univariate polynomial does not exceed the number of sign changes in its sequence of coefficients. Moreover, it has the same parity.

Theorem [Sturm].
The number of zeros of a real univariate polynomial $p$ on the interval $(a, b]$ is given by $V(a) - V(b)$, with $V()$ the number of sign changes in its Sturm sequence $p, p_1, p_2, \ldots$. 
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The number of zeros of a real univariate polynomial $p$ on the interval $(a, b]$ is given by $V(a) - V(b)$, with $V()$ the number of sign changes in its Sturm sequence $p, p_1, p_2, \ldots$. 
Starting from \( f_0 := p \), \( f_1 := q - (b_0/a_0)p \), form the Euclidean algorithm sequence

\[
f_{j-1} = q_j f_j + f_{j+1}, \quad j = 1, \ldots, k, \quad f_{k+1} = 0.
\]

Then \( f_k \) is the greatest common divisor of \( p \) and \( q \). This gives a continued fraction representation

\[
R(z) = \frac{f_1(z)}{f_0(z)} = \frac{1}{q_1(z) + \frac{1}{q_2(z) + \frac{1}{q_3(z) + \cdots + \frac{1}{q_k(z)}}}}.
\]
Euclidean algorithm and continued fractions

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Generalized Jacobi matrices

\[ J(z) := \begin{bmatrix}
q_k(z) & -1 & 0 & \ldots & 0 & 0 \\
1 & q_{k-1}(z) & -1 & \ldots & 0 & 0 \\
0 & 1 & q_{k-2}(z) & \ldots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \ldots & q_2(z) & -1 \\
0 & 0 & 0 & \ldots & 1 & q_1(z)
\end{bmatrix}. \]

Remark 1. \( h_j(z) := f_j(z)/f_k(z) \) is the leading principal minor of \( J(z) \) of order \( k - j \). In particular, \( h_0(z) = \det J(z) \).

Remark 2. Eigenvalues of the generalized eigenvalue problem

\[ J(z)u = 0 \]

are closely related to properties of \( R(z) \).
**Generalized Jacobi matrices**

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are closely related to properties of \( R(z) \).
In the regular case,

\[ q_j(z) = \alpha_j z + \beta_j, \quad \alpha_j, \beta_j \in \mathbb{C}, \alpha_j \neq 0. \]

The polynomials \( f_j \) satisfy the three-term recurrence relation

\[ f_{j-1}(z) = (\alpha_j z + \beta_j)f_j(z) + f_{j+1}(z), \quad j = 1, \ldots, r. \]

\[ R(z) = \frac{f_1(z)}{f_0(z)} = \frac{1}{\alpha_1 z + \beta_1 + \frac{1}{\alpha_2 z + \beta_2 + \frac{1}{\alpha_3 z + \beta_3 + \cdots + \frac{1}{\alpha_r z + \beta_r}}}}. \]
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In the doubly regular case, 

\[ q_{2j}(z) = c_{2j}, \quad j = 1, \ldots, \left\lfloor \frac{k}{2} \right\rfloor, \]

\[ q_{2j-1}(z) = c_{2j-1}z, \quad j = 1, \ldots, r. \]

\[ R(z) = \frac{f_1(z)}{f_0(z)} = \frac{1}{c_1z + \frac{1}{c_2 + \frac{1}{c_3z + \cdots + \frac{1}{T}}}}, \text{ where } \]

\[ T := \begin{cases} 
  c_{2r} & \text{if } |R(0)| < \infty, \\
  c_{2r-1}z & \text{if } R(0) = \infty.
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\end{cases} \]
Sturm’s algorithm is a variation of the Euclidean algorithm

\[ f_{j-1}(z) = q_j(z)f_j(z) - f_{j+1}(z), \quad j = 0, 1, \ldots, k, \]

where \( f_{k+1}(z) = 0 \). The polynomial \( f_k \) is the greatest common divisor of \( p \) and \( q \).

The Sturm algorithm is regular if the polynomials \( q_j \) are linear.

Theorem [Sturm].

\[ \text{Ind}_{-\infty}^{\infty} \left( \frac{f_1}{f_0} \right) = n - 2V(h_0, \ldots, h_n) \text{ where } h_k \text{ is the leading coefficient of } f_k. \]
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Definition. 

\[ \text{Ind}_\omega(F) := \begin{cases} 
+1, & \text{if } F(\omega - 0) < 0 < F(\omega + 0), \\
-1, & \text{if } F(\omega - 0) > 0 > F(\omega + 0), 
\end{cases} \]

is the index of the function \( F \) at its real pole \( \omega \) of odd order.

Theorem [Gantmacher].

If a rational function \( R \) with exactly \( r \) poles is represented by a series

\[ R(z) = s_{-1} + \frac{s_0}{z} + \frac{s_1}{z^2} + \cdots, \]

then

\[ \text{Ind}_{-\infty}^{+\infty} = r - 2V(D_0(R), D_1(R), D_2(R), \ldots, D_r(R)). \]
Cauchy indices

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Let $R(z)$ be a rational function expanded in its Laurent series at $\infty$

$$R(z) = s_{-1} + \frac{s_0}{z} + \frac{s_1}{z^2} + \frac{s_2}{z^3} + \cdots.$$ 

Introduce the infinite Hankel matrix $S := [s_{i+j}]_{i,j=0}^\infty$ and consider the leading principal minors of $S$:

$$D_j(S) := \det \begin{bmatrix} s_0 & s_1 & s_2 & \ldots & s_{j-1} \\ s_1 & s_2 & s_3 & \ldots & s_j \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ s_{j-1} & s_j & s_{j+1} & \ldots & s_{2j-2} \end{bmatrix}, \quad j = 1, 2, 3, \ldots.$$ 

These are Hankel minors or Hankel determinants.
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These are Hankel minors or Hankel determinants.
Let $R(z) = \frac{q(z)}{p(z)}$, $p(z) = a_0 z^n + \cdots + a_n$, $a_0 \neq 0$, $q(z) = b_0 z^n + \cdots + b_n$.

For each $j = 1, 2, \ldots$, denote

\[
\nabla_{2j}(p, q) := \det \begin{bmatrix}
  a_0 & a_1 & a_2 & \ldots & a_{j-1} & a_j & \ldots & a_{2j-1} \\
  b_0 & b_1 & b_2 & \ldots & b_{j-1} & b_j & \ldots & b_{2j-1} \\
  0 & a_0 & a_1 & \ldots & a_{j-2} & a_{j-1} & \ldots & a_{2j-2} \\
  0 & b_0 & b_1 & \ldots & b_{j-2} & b_{j-1} & \ldots & b_{2j-2} \\
  \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
  0 & 0 & 0 & \ldots & a_0 & a_1 & \ldots & a_j \\
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\end{bmatrix}.
\]

These are the **Hurwitz minors** or **Hurwitz determinants**.

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Theorem [Hurwitz].
Let $R(z) = q(z)/p(z)$ with notation as above. Then
\[ \nabla_2 j(p, q) = a_0^{2j} D_j(R), \quad j = 1, 2, \ldots. \]

Corollary.
Let $T(z) = -1/R(z)$ with notation as above. Then
\[ D_j(S) = s_{-1}^{2j} D_j(T), \quad j = 1, 2, \ldots. \]
Theorem [Hurwitz].

Let \( R(z) = q(z)/p(z) \) with notation as above. Then

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Corollary.

Let \( T(z) = -1/R(z) \) with notation as above. Then

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D_j(S) = s_{-1}^{2j} D_j(T), \quad j = 1, 2, \ldots.
\]
Let $p$ be a real polynomial

$$p(z) = a_0 z^n + a_1 z^{n-1} + \ldots + a_n =: p_0(z^2) + zp_1(z^2), \quad a_0 > 0, \ a_i \in \mathbb{R},$$

Let $n = \deg p$ and $m = \left\lfloor \frac{n}{2} \right\rfloor$.

for $n = 2m$

$$p_0(u) = a_0 u^m + a_2 u^{m-1} + \ldots + a_{n-2} u + a_n,$$
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$$p_1(u) = a_0u^m + a_2u^{m-1} + \ldots + a_{n-3}u + a_{n-1}.$$
Introduce the function associated with the polynomial $p(z) = p_0(z^2) + zp_1(z^2)$

$$\Phi(u) = \frac{p_1(u)}{p_0(u)}$$

**Definition.**
A polynomial is Hurwitz stable if all its zeros lie in the open left half-plane.

**Main Theorem of Stability**
A polynomial $z \mapsto p(z) = p_0(z^2) + zp_1(z^2)$ is Hurwitz stable if and only if

$$\Phi(u) = \beta + \sum_{j=1}^{m} \frac{\alpha_j}{u + \omega_j}, \quad \beta \geq 0, \ \alpha_j, \omega_j > 0, \ m = \left\lfloor \frac{n}{2} \right\rfloor.$$
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Hurwitz and Lienard–Chipart Theorems

**Hurwitz Theorem**
A polynomial $p$ of degree $n$ is Hurwitz stable if and only if

$$
\Delta_1(p) > 0, \ \Delta_2(p) > 0, \ldots, \ \Delta_n(p) > 0.
$$

**Lienard and Chipart Theorem**
A polynomial $p$ of degree $n$ is Hurwitz stable if and only if

$$
\Delta_{n-1}(p) > 0, \ \Delta_{n-3}(p) > 0, \ \Delta_{n-5}(p) > 0, \ldots
$$

and

$$
a_n > 0, \ a_{n-2} > 0, \ a_{n-4} > 0, \ldots
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Hurwitz and Lienard–Chipart Theorems

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$$a_n > 0, a_{n-2} > 0, a_{n-4} > 0, \ldots$$
Using properties of functions mapping the UHP to LoHP and the main theorem of stability, one can obtain

**Stieltjes criterion of stability**

A polynomial $p$ of degree $n$ is Hurwitz stable if and only if its associated function $\Phi$ has the following Stieltjes continued fraction expansion

$$\Phi(u) = \frac{p_1(u)}{p_0(u)} = c_0 + \frac{1}{c_1 u + \frac{1}{c_2 u + \frac{1}{\ddots + \frac{1}{c_{2m} u}}}}$$

where $c_0 \geq 0$ and $c_i > 0$, $i = 1, \ldots, 2m$, and $m = \left\lfloor \frac{n}{2} \right\rfloor$. 

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Further criteria of stability

Theorem.
A polynomial $p$ of degree $n$ is Hurwitz stable if and only if the infinite Hankel matrix $S = \left\| s_{i+j} \right\|_{i,j=0}^{\infty}$ is a sign-regular matrix of rank $m$, where $m = \left\lfloor \frac{n}{2} \right\rfloor$.

Theorem.
A polynomial $f = p(z^2) + zq(z^2)$ is stable if and only if its infinite Hurwitz matrix $H(p, q)$ is totally nonnegative.
Further criteria of stability

**Theorem.**
A polynomial $p$ of degree $n$ is Hurwitz stable if and only if the infinite Hankel matrix $S = \|s_{i+j}\|_{i,j=0}^\infty$ is a sign-regular matrix of rank $m$, where $m = \left\lfloor \frac{n}{2} \right\rfloor$.

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Theorem

If \( g(z) = g_0 z^l + g_1 z^{l-1} + \ldots + g_l \), then

\[
H(p \cdot g, q \cdot g) = H(p, q) \mathcal{I}(g),
\]

where

\[
\mathcal{I}(g) := \begin{bmatrix}
g_0 & g_1 & g_2 & g_3 & g_4 & \cdots \\
0 & g_0 & g_1 & g_2 & g_3 & \cdots \\
0 & 0 & g_0 & g_1 & g_2 & \cdots \\
0 & 0 & 0 & g_0 & g_1 & \cdots \\
0 & 0 & 0 & 0 & g_0 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{bmatrix}.
\]

Here we set \( g_i = 0 \) for all \( i > l \).
Another factorization

**Theorem**

If the Euclidean algorithm for the pair $p, q$ is doubly regular, then $H(p, q)$ factors as

$$H(p, q) = J(c_1) \cdots J(c_k) H(0, 1) T(g),$$

where

$$J(c) := \begin{bmatrix} c & 1 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 1 & 0 & 0 & \cdots \\ 0 & 0 & c & 1 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 1 & \cdots \\ 0 & 0 & 0 & 0 & c & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

and

$$H(0, 1) = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 1 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$
Stable polynomials and Stieltjes continued fractions

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where $c_0 \geq 0$ and $c_i > 0$, $i = 1, \ldots, 2m$, and $m = \left\lfloor \frac{n}{2} \right\rfloor$. 

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"Jacobi" criterion of stability

A polynomial \( p \) of degree \( n \) is Hurwitz stable if and only if its associated function \( \Phi \) has the following Jacobi continued fraction expansion

\[
\Phi(u) = \frac{p_1(u)}{p_0(u)} = -\alpha u + \beta + \frac{1}{\frac{\alpha_1 u + \beta_1}{1} - \frac{1}{\frac{\alpha_2 u + \beta_2}{1} + \cdots + \frac{1}{\alpha_n u + \beta_n}}},
\]

where \( \alpha \geq 0, \alpha_j > 0 \) and \( \beta, \beta_j \in \mathbb{R} \).
Other topics

- Stieltjes, Jacobi, other continued fractions. Padé approximation
- Orthogonal polynomials. Moment problems
- Nevanlinna functions
- Pólya frequency sequences and functions
- Laguerre-Pólya class and its generalizations
- Total nonnegativity
- Matrix factorizations
- Hurwitz rational and meromorphic functions
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Olga Holtz

Zero localization: from 17th-century algebra to challenges of today
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