How to find roots of polynomials — and how university students can do successful research on a century-old problem

Dierk Schleicher

Jacobs University Bremen

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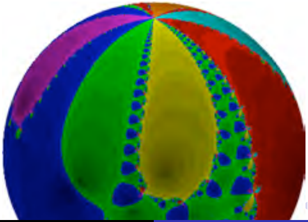


Overview

- Polynomials and their roots (real and complex numbers)
- Newton's method for root-finding
- Classical questions and modern answers by university students
- An efficient algorithm by a high school student that beats the professionals

... and various examples where students can make a difference in relevant current research problems!

3 Polynomials and Their Roots

A *polynomial* of degree d is a map $p(x) = a_d x^d + a_{d-1} x^{d-1} + \cdots + a_1 x + a_0$, where all a_n are constant coefficients and x is the variable.

A *root* of p is a number α with $p(\alpha) = 0$.

Every root yields a linear factor by long division: if $p(\alpha) = 0$, then $p(x) = (x - \alpha) \cdot p_1(x)$, where $p_1(x)$ is a polynomial of degree $d - 1$. Can be repeated inductively.

Over the reals, not every polynomial has a root: for $p(x) = x^2 + 1$, no $x \in \mathbb{R}$ satisfies $p(x) = 0$.

For this purpose, complex numbers were invented: i is a number with $i^2 = -1$; then $p(i) = 0$.

Theorem 1 (The Fundamental Theorem of (for?) Algebra)

Every polynomial p of degree d completely factors into d linear factors over $\mathbb{C}$: $p(z) = a_d (z - \alpha_1) \cdot (z - \alpha_2) \cdot \cdots \cdot (z - \alpha_d)$.

4 Multiplication and Factorization

Can one convert between the coefficient form $p(z) = a_d z^d + a_{d-1} z^{d-1} + \cdots + a_1 z + a_0$ and the factorized form $p(z) = a_d(z - \alpha_1) \cdot (z - \alpha_2) \cdot \cdots \cdot (z - \alpha_d)$?

Which way is easier?

From factorized form, can "easily" multiply out: simple algorithm. May be a tedious: k-th coefficient is $\pm$ sum of products of all k-tuples of roots! There are many such k-tuples!

Conversely, how does one find the roots from the coefficients?

$\boxed{d = 2}$ $p(x) = ax^2 + bx + c$ has roots $-b/2a \pm \sqrt{b^2 - 4ac}/2a$.

$\boxed{d = 3}$ *Cardano formula:* if $p(x) = ax^3 + bx^2 + cx + d$, then set

$$\Delta := \frac{27a^2d^2 + 4b^3d - 18abcd + 4ac^3 - b^2c^2}{108a^4},$$

and one root of p is

$$x = \sqrt[3]{\frac{2b^3 - 9abc + 27a^2d}{54a^2} + \sqrt{\Delta}} + \sqrt[3]{\frac{2b^3 - 9abc - 27a^2d}{54a^2} - \sqrt{\Delta}}$$

5 Cardano Formulas for Degree $d = 4$

$\boxed{d = 4}$ *Cardano* again: if $p(x) = ax^4 + bx^3 + cx^2 + dx + e$, then

$$\alpha = -3b^2/8a^2 + c/a$$
$$\beta = b^3/8a^3 - bc/2a^2 + d/a$$
$$\gamma = -3b^4/256a^4 + b^2c/16a^3 - bd/4a^2 + e/a$$
$$p = -\alpha^2/12 - \gamma$$
$$q = -\alpha^3/108 + \alpha\gamma/3 - \beta^2/8$$
$$u = \sqrt[3]{-q/2 + \sqrt{q^2/4 + p^3/27}}$$
$$y = -5\alpha/6 + u - p/3u$$
$$w = \sqrt{\alpha + 2y}$$

and the four roots of p are

$$x = -\frac{b}{4a} + \frac{1}{2}\left(\pm w \pm \sqrt{-3\alpha - 2y - \mp 2\beta/w}\right)$$

6 Formulas for Higher Degrees?

Theorem 2 (Ruffini-Abel Theorem; Galois Theory)

There cannot be such formulas for $d \geq 5$ in general.

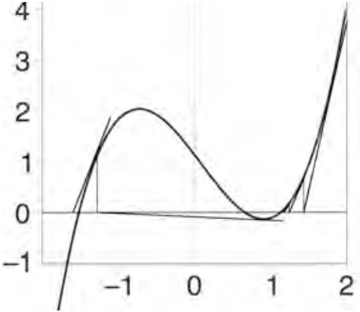
Need to find roots by approximation methods!

Question: Given a differentiable $p\colon \mathbb{R} \to \mathbb{R}$, how can we find the zeroes of p?

Newton Method (oooold!): Iterate $N_p\colon x \mapsto x - p(x)/p'(x)$.

Advantage:

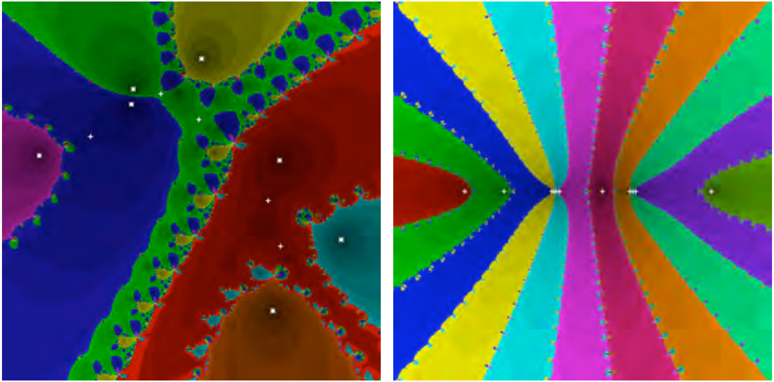
- locally fast convergence (quadratic!)
- easy to implement
- robust method
- parallel computation possible



Problems: Global convergence and dynamics poorly understood

Numerical analysts: "...it is well known that the [Newton] iteration may behave in a *chaotic* way! — Don't use Newton!"

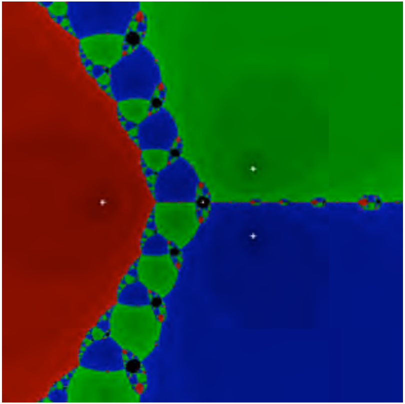
If experts agree: why should we work on Newton? What progress can students make centuries later?



The formula $N_p(z) = z - p(z)/p'(z)$ works over $\mathbb{C}$: for polynomial p the Newton method is the iteration of a rational map N_p. Colors distinguish basins of roots. Right: complex basins help even when roots are all real.

Problems of the Newton Method

- ⋆ bad starting points (near poles jump near ∞)
- ⋆ basin boundaries may have positive measure!
- ⋆ periodic basins(!) right: in black for $z^3 - 2z + 2$
- ⋆ global dynamics hard to control
- ⋆ find *all* roots without deflation!
- ⋆ how many iterations needed?



Systematic computer experiments give new ideas

Original motivation: understand dynamics, not fast numerics

10 Good Starting Points for the Newton Method

Let P_d be the space of all complex polynomials of degree d, normalized so that all roots within unit disk $\mathbb{D}$

Theorem 1 (Good Starting Points; Hubbard-S-Sutherland 2001)

For every degree d there is an explicit universal set S_d of starting points so that for *every* $p \in P_d$ and *every* root α of p at least one $z_0 \in S_d$ converges to α under iteration of N_p.
The set S_d has small cardinality $|S_d| \approx 1.1 d(\log d)^2$.
If all roots are real, then $1.3d$ starting points suffice.

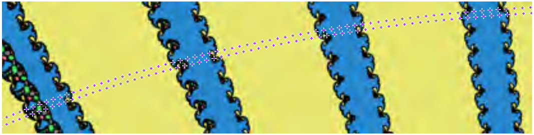
(First paper with color pictures in *Inventiones*.)

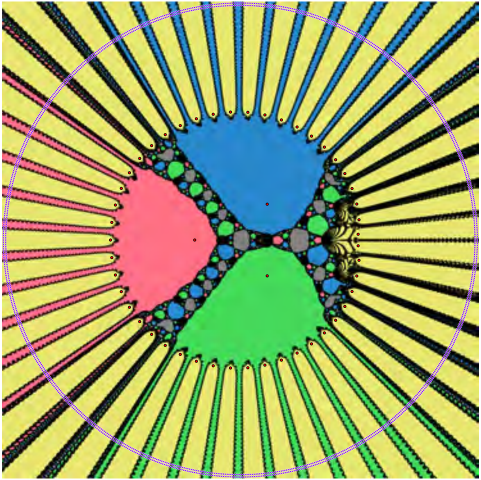
The points in S_d are on $\log d$ circles with each $d \log d$ points.

Theorem 2 (Probabilistic. — Bollobás, Lackmann, S. 2011)

Probabilistically, $|S_d| = \mathit{const} \cdot d(\log\log d)^2$ points suffice.

11 The Guaranteed Set of Starting Points





A polynomial of degree 50: 47 roots are on the unit circle, three are in the disk (with basins in red, blue, green).

Our guaranteed set of starting points S_d is drawn to scale.

Note the grey domains where Newton's method does not converge to any root.

John Hubbard: my PhD advisor, professor at Cornell University/NY, USA

Belá Bollobás: first-ever IMO champion 1959–, professor in Memphis/USA and Cambridge/UK, guest of honor at IMO 2009

Malte Lackmann: participant IMO 2009, then guest at Jacobs University — and now has Erdös number $2 - 2\varepsilon$!

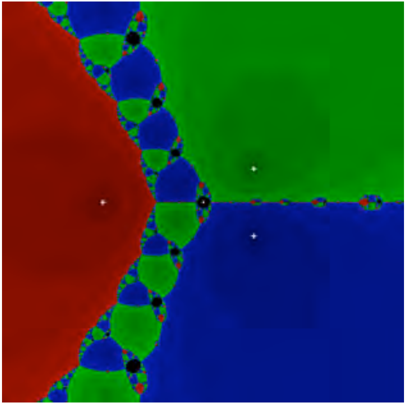


Belá, Malte and I met at the 50th IMO anniversary 2009 at Jacobs University in Bremen

The 50th IMO anniversary, July 2009, with guests of honor Belá Bollobás, Timothy Gowers, László Lovász, Stanislav Smirnov, Terence Tao, Jean-Christophe Yoccoz — 4 Fields medalists and 565 future math experts!

Problem: There are Newton maps with extra cycles of higher periods, hence open sets of points not converging to any roots.

Question (Smale): Give classification of all such polynomials!



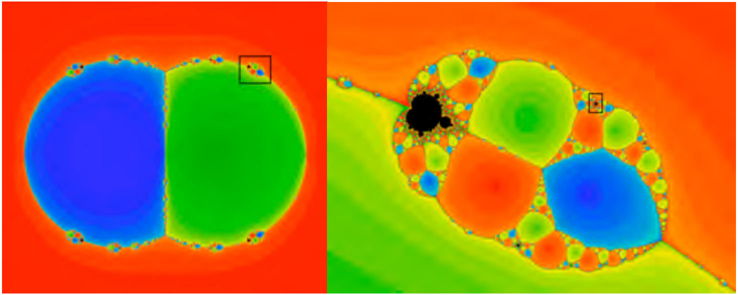
Simplest case:
$p(z) = z^3 - 2z + 2$

15 The Space of All Cubic Newton Maps

Simplest non-trivial case: the space of *all* cubic polynomials: each point is a different polynomial.

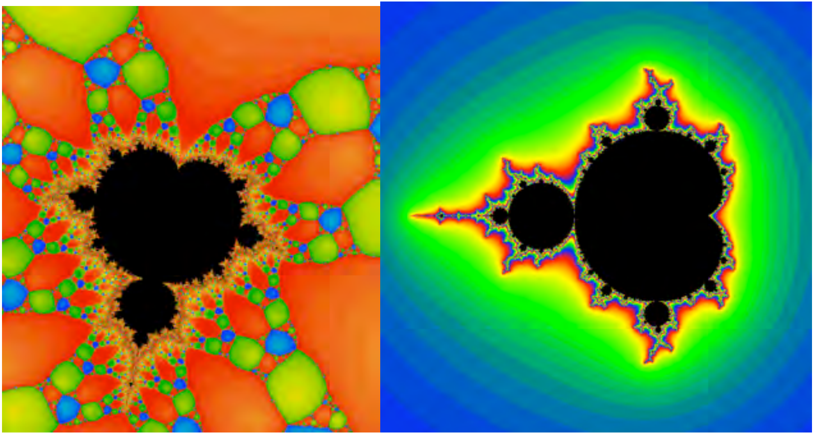
Translate coordinates so that one root is at $z = 0$, then scale so that another one is at $z = 1$; third root then at $\lambda \in \mathbb{C}$.

Write $p(z) = z(z - 1)(z - \lambda)$ with $\lambda \in \mathbb{C}$



Black: polynomials with attracting cycles of higher periods

16 Cubic Newton Map with Periodic Basins



All extra attracting basins are classified by homeomorphic copies of the **Mandelbrot set:** the space of iterated quadratic polynomials.

17 Classification of Newton Maps as Dynamical Systems

PhD Thesis Yauhen “Zhenya” Mikulich, Jacobs U Bremen 2011

Theorem 3 (Newton Map Classification. — Mikulich 2011)

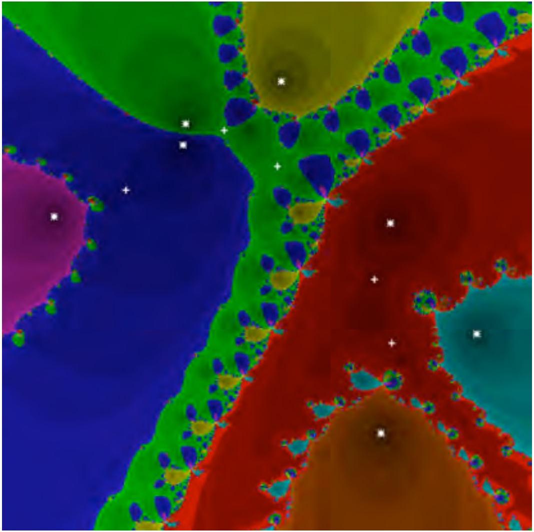
Classification of *all* postcritically finite Newton maps of *all* degrees. Includes answer to Smale’s question: classify those with open sets of non-converging points, hence attracting cycles of higher periods.

Provides second class of dynamically classified rational maps (after polynomials).

Classification in terms of finite graphs that describe the basins of the various roots and how they are connected.

Project completed 2014: two preprints by Mikulich, Lodge, S.

A Typical Picture.



Starting point and motivation: typical dynamical plane of the Newton method (here: for polynomial of degree 7).

19 Newton as an Efficient Algorithm

... but how many iterations are required?

Theorem 4 (Fast Convergence. — S. 2002/2006/2011/2013)

For every degree d and every $\varepsilon > 0$ there is an $M_d > 0$ so that for each $p \in P_d$ and each root α_j of p, at least one starting point $z_j \in S_d$ converges to α_j under N_p, and $|N_p^{\circ m_j}(z_j) - \alpha_j| < \varepsilon$ with $\sum_j m_j < M_d$.
We have $M_d \in O(d^4 + d \log \varepsilon)$ in the worst case, and $M_d \in O(d^2 \log^4 d + d \log|\log \varepsilon|)$ in the expected case.

What this means — development over time:

- 2002 S.: existence of explicit upper bound on number of iterations: at most $\approx 4^d$ (uniform bound but very slow)
- 2006 S.: improvement of upper bound: at most $\approx d^6$
- 2011 S.: further improvement to at most $\approx d^3$ iterations
- 2013, Bachelor thesis of Todor Bilarev (Jacobs Univ): at most $d^2 \log^4 d$ iterations (very close to optimum d^2)

Robin Stoll (18 years), high school graduate 2014

"how can I find all roots on my computer for degree d"

Specific polynomials of degree $2^{20} = 1\,048\,576 > 10^6$

- December 2014: found all roots in 5 weeks computer time
- January 2015: improvements in compiler: 30 hours computer time
- March 2015: improvements in algorithm: 10 minutes computer time

And what do the experts say? — MPSolve 3.0 professional package, specific software polynomial degree 2^{24}: takes 5 months on their high-speed computers

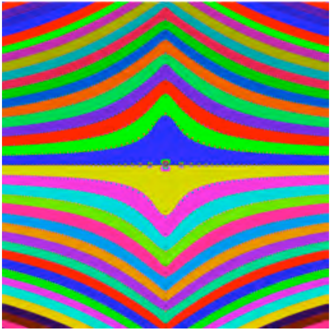
Robin, on his home computer: 8998 seconds!

21 Everything Solved? And What Next?

Is everything solved now? Can currently active students still do research?

If questions remained open for 300 years, should they end just now?

Find zeroes of the Riemann ζ function by Newton's method!
Needs active and talented students
(high school graduates, undergraduates $\to$ Bachelor thesis, PhD students!)



There are plenty of research opportunities in mathematics waiting to be solved today! Find yourself a place and opportunity where this happens! — At this summer school, at this university, or elsewhere.

Thank you for your attention!

I look forward to meeting you again.

dierk@jacobs-university.de