Endomorphisms of Weyl Algebras and the Jacobian Conjecture

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The Jacobian Conjecture (JC\(_n\)) asks:

*Is any polynomial endomorphism of \(\mathbb{C}^n\), \(F : x_i \mapsto F_i(\vec{x})\) with constant Jacobian \(\det\left(\frac{\partial F_i}{\partial x_j}\right)\) an automorphism?*

The Dixmier Conjecture (DC\(_n\)) asks:

*Is any endomorphism of the Weyl Algebra \(W_n\) of differential operators, \(W_n = \mathbb{C}\left[x_1, \ldots, x_n, \frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}\right]\), an automorphism?*

It is well-known that DC\(_n\) \(\implies\) JC\(_n\). Ten years ago, it was proved that JC\(_2\) \(\implies\) DC\(_1\). Recently, Kontsevich and I proved that JC\(_{2n}\) \(\implies\) DC\(_n\) for all \(n\).

The proof is based on reduction mod \(p\). After this reduction, \(W_n \otimes \mathbb{Z}_p\) becomes a finite-dimensional algebra over its center, and any endomorphism of \(W_n \otimes \mathbb{Z}_p\) is uniquely defined by its restriction to the center of \(W_n \otimes \mathbb{Z}_p\).

This approach is related to Kontsevich’s conjecture that Aut\(W_n\) is isomorphic to the group of symplectomorphisms of \(\mathbb{C}^{2n}\). For \(n > 1\), this Kontsevich Conjecture is still open.