Endomorphisms of Weyl Algebras and the Jacobian Conjecture

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The *Jacobian Conjecture* (JC_n) asks:

> *Is any polynomial endomorphism of* $\mathbb{C}^n$*,* $F : x_i \mapsto F_i(\vec{x})$ *with constant Jacobian* $\det\left(\frac{\partial F_i}{\partial x_j}\right)$ *an automorphism?*

The *Dixmier Conjecture* (DC_n) asks:

> *Is any endomorphism of the Weyl Algebra* W_n *of differential operators,*
>
> $$W_n = \mathbb{C}\left[x_1, \ldots, x_n, \frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}\right],$$
>
> *an automorphism?*

It is well-known that $\mathrm{DC}_n \Longrightarrow \mathrm{JC}_n$. Ten years ago, it was proved that $\mathrm{JC}_2 \Longrightarrow \mathrm{DC}_1$. Recently, Kontsevich and I proved that $\mathrm{JC}_{2n} \Longrightarrow \mathrm{DC}_n$ for all n.

The proof is based on reduction mod p. After this reduction, $W_n \otimes \mathbb{Z}_p$ becomes a finite-dimensional algebra over its center, and any endomorphism of $W_n \otimes \mathbb{Z}_p$ is uniquely defined by its restriction to the center of $W_n \otimes \mathbb{Z}_p$.

This approach is related to Kontsevich's conjecture that $\operatorname{Aut} W_n$ is isomorphic to the group of symplectomorphisms of $\mathbb{C}^{2n}$. For $n > 1$, this Kontsevich Conjecture is still open.