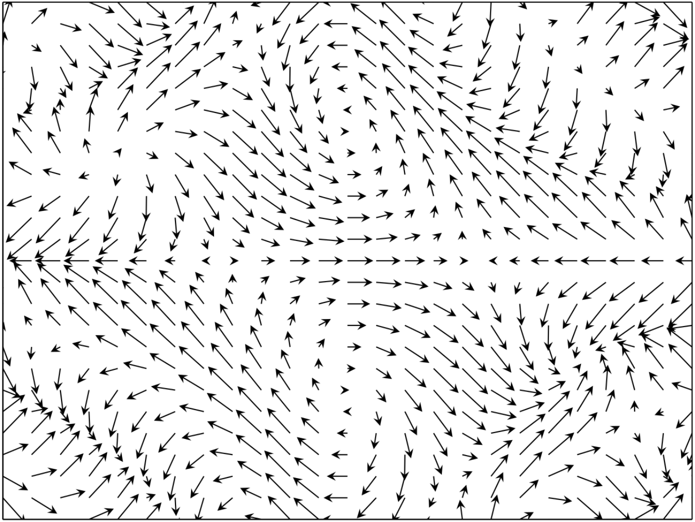
Covariant Lyapunov vectors
-
Linear Modes for Turbulent Background Flow

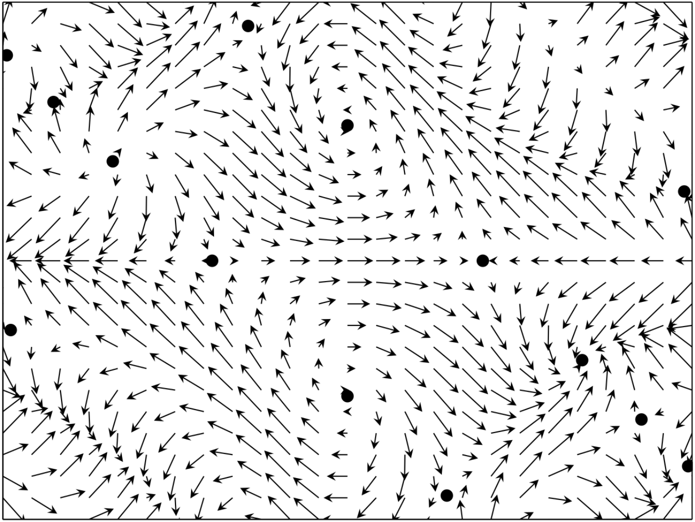
Florian Noethen

June 12, 2020

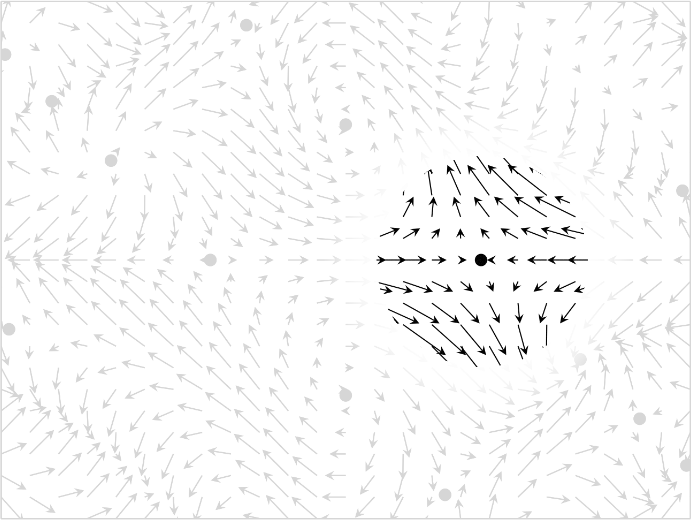
Nonlinear vectorfield



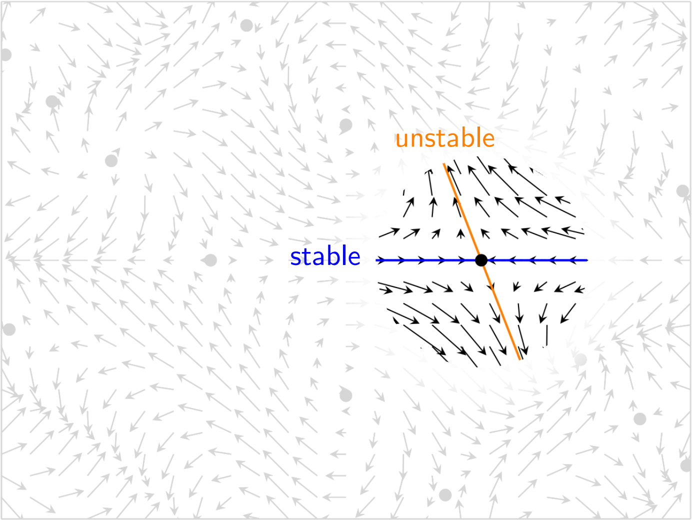
Stationary points



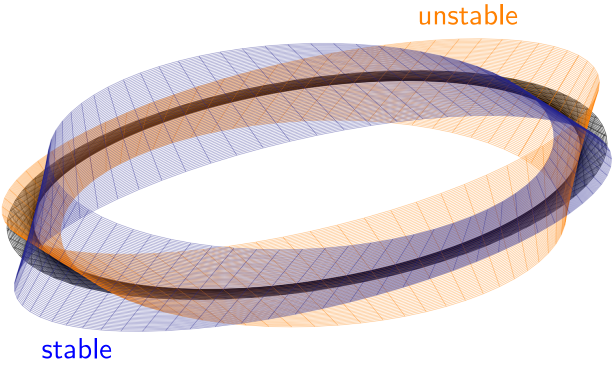
Linearization



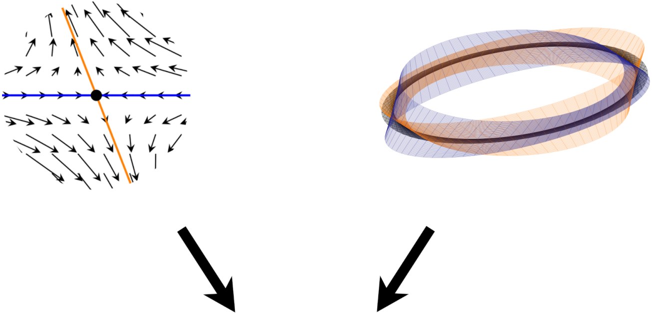
Linearization



Periodic orbit

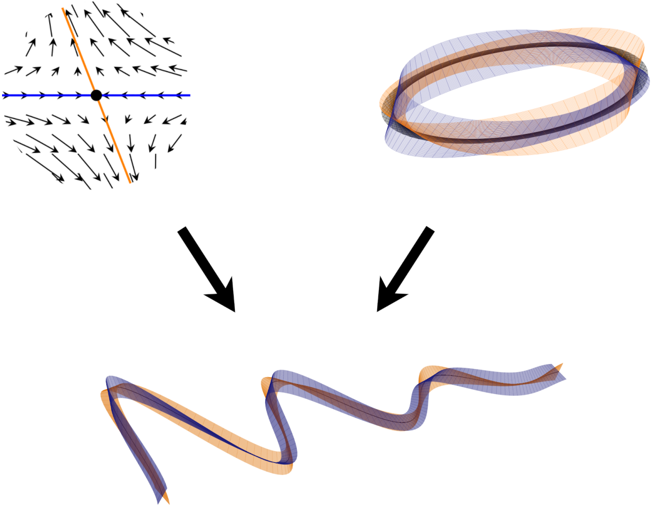


What about other trajectories?

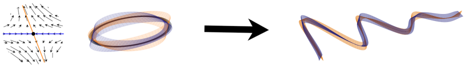


?

What about other trajectories?



Multiplicative Ergodic Theorem (MET)[1,2,3]



growth rates	eigenvalues / Floquet exponents	Lyapunov exponents
directions	eigenvectors	Covariant Lyapunov Vectors (CLVs)
spaces	eigenspaces	Oseledets spaces

[1] V. I. Oseledets. "A multiplicative ergodic theorem. Characteristic Ljapunov, exponents of dynamical systems". In: *Transactions of the Moscow Mathematical Society* 19 (1968), pp. 197–231.

[2] Ludwig Arnold. *Random Dynamical Systems*. Springer monographs in mathematics. Berlin and Heidelberg: Springer, 1998. ISBN: 9783642083556. DOI: 10.1007/978-3-662-12878-7.

[3] Cecilia González-Tokman and Anthony Quas. "A semi-invertible operator Oseledets theorem". In: *Ergodic Theory and Dynamical Systems* 34.4 (2014), pp. 1230–1272. ISSN: 0143-3857. DOI: 10.1017/etds.2012.189.

Some applications

- sensitivity analysis/statistics[4][5]
- mode reduction and decoupling of tangent space[6]
- coherent structures (slow mixing sets)[7][8]

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[4] Sebastian Schubert and Valerio Lucarini. "Covariant Lyapunov vectors of a quasi-geostrophic baroclinic model: analysis of instabilities and feedbacks". In: *Quarterly Journal of the Royal Meteorological Society* 141.693 (2015), pp. 3040–3055. ISSN: 00359009. DOI: 10.1002/qj.2588.

[5] Angxiu Ni and Qiqi Wang. "Sensitivity analysis on chaotic dynamical systems by Non-Intrusive Least Squares Shadowing (NILSS)". In: *Journal of Computational Physics* 347 (2017), pp. 56–77. ISSN: 00219991. DOI: 10.1016/j.jcp.2017.06.033.

[6] Kazumasa A. Takeuchi et al. "Hyperbolic decoupling of tangent space and effective dimension of dissipative systems". In: *Physical Review E* 84.4 (2011), p. 046214. DOI: 10.1103/PhysRevE.84.046214.

[7] Cecilia González-Tokman. "Multiplicative ergodic theorems for transfer operators: Towards the identification and analysis of coherent structures in non-autonomous dynamical systems". In: *Contributions of Mexican mathematicians abroad in pure and applied mathematics*. Ed. by Fernando Galaz-García, J. C. P. Millán, and Pedro Solórzano. Vol. 709. Contemporary Mathematics. Guanajuato, Mexico: American Mathematical Society, 2018, pp. 31–52. ISBN: 9781470442866. DOI: 10.1090/conm/709/14290.

[8] Chantelle Blachut and Cecilia González-Tokman. "A tale of two vortices: how numerical ergodic theory and transfer operators reveal fundamental changes to coherent structures in non-autonomous dynamical systems". In: (2020). arXiv: 2003.07559v2. URL: https://arxiv.org/pdf/2003.07559.

How do we compute CLVs?

SVD, Ginelli, Wolfe-Samelson, Kuptov-Parlitz, ...[9]

[9] Gary Froyland et al. "Computing covariant Lyapunov vectors, Oseledets vectors, and dichotomy projectors: A comparative numerical study". In: *Physica D: Nonlinear Phenomena* 247.1 (2013), pp. 18–39. ISSN: 01672789. DOI: 10.1016/j.physd.2012.12.005.

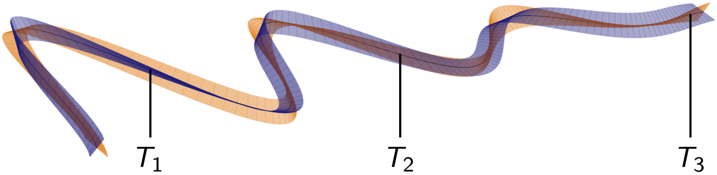
How do we compute CLVs?

SVD, Ginelli, Wolfe-Samelson, Kuptov-Parlitz, ...[10]

- dynamical algorithm
- computes Lyapunov exponents and CLVs
- no additional assumptions aside from MET

[10] Gary Froyland et al. "Computing covariant Lyapunov vectors, Oseledets vectors, and dichotomy projectors: A comparative numerical study". In: *Physica D: Nonlinear Phenomena* 247.1 (2013), pp. 18–39. ISSN: 01672789. DOI: 10.1016/j.physd.2012.12.005.

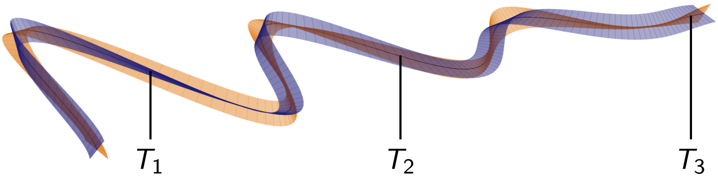
Ginelli's algorithm[11]



1. forward propagation
   1.1 initial vectors
   1.2 propagate from past to present
   1.3 propagate from present to future

[11] F. Ginelli et al. "Characterizing Dynamics with Covariant Lyapunov Vectors". In: *Physical review letters* 99.13 (2007), p. 130601. ISSN: 0031-9007. DOI: 10.1103/PhysRevLett.99.130601.

Ginelli's algorithm[11]



1. forward propagation
   1.1 initial vectors
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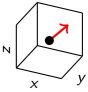
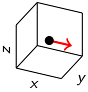
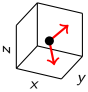
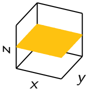
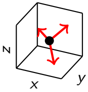
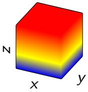
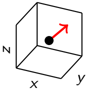
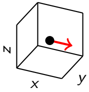
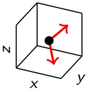
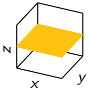
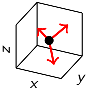
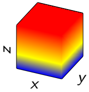
2. backward propagation
   2.1 initial vectors subject to forward propagated vectors
   2.2 propagate from future to present

[11] F. Ginelli et al. "Characterizing Dynamics with Covariant Lyapunov Vectors". In: *Physical review letters* 99.13 (2007), p. 130601. ISSN: 0031-9007. DOI: 10.1103/PhysRevLett.99.130601.

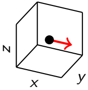
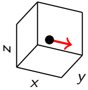
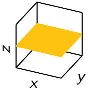
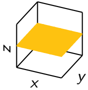
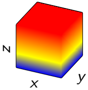
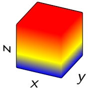
Forward propagation - step 1.2

$t = T_1$	$t = T_2$	good approx. to
		CLV 1
		CLV 1 $\oplus$ CLV 2
		CLV 1 $\oplus$ CLV 2 $\oplus$ CLV 3

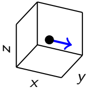
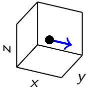
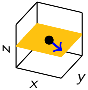
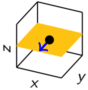
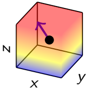
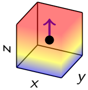
Forward propagation - step 1.2

$t = T_1$	$t = T_2$	good approx. to
		CLV 1
		CLV 1 $\oplus$ CLV 2
		CLV 1 $\oplus$ CLV 2 $\oplus$ CLV 3

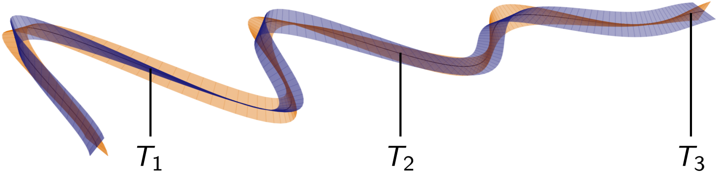
Forward propagation - step 1.3

$t = T_2$	$t = T_3$	good approx. to
		CLV 1
		CLV 1 $\oplus$ CLV 2
		CLV 1 $\oplus$ CLV 2 $\oplus$ CLV 3

Backward propagation - step 2.2

$t = T_3$	$t = T_2$	good approx. to
		CLV 1
		CLV 2
		CLV 3

Ginelli's algorithm[12]



1 forward propagation

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My research

Main questions

Does Ginelli's algorithm converge?

What is the speed of converge?

What happens in different settings?

Main results

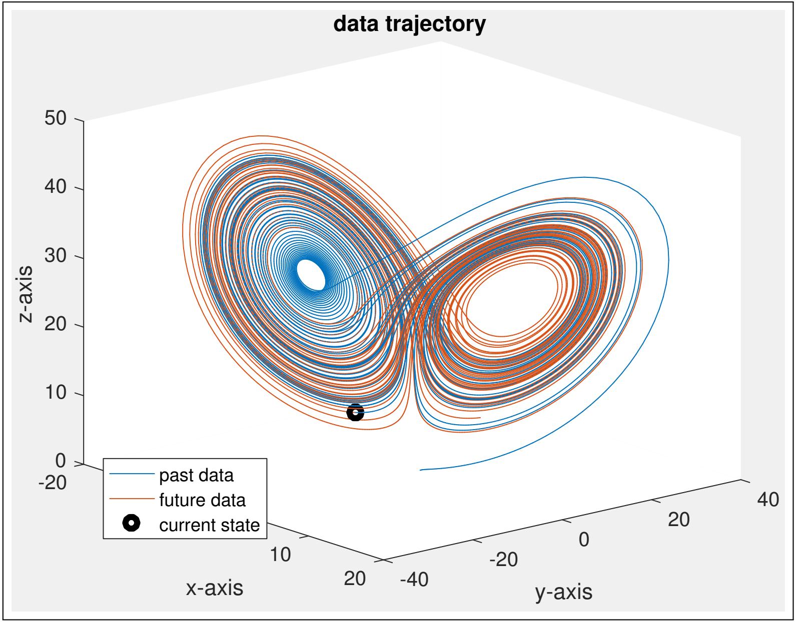
Convergence theorems in finite dim.

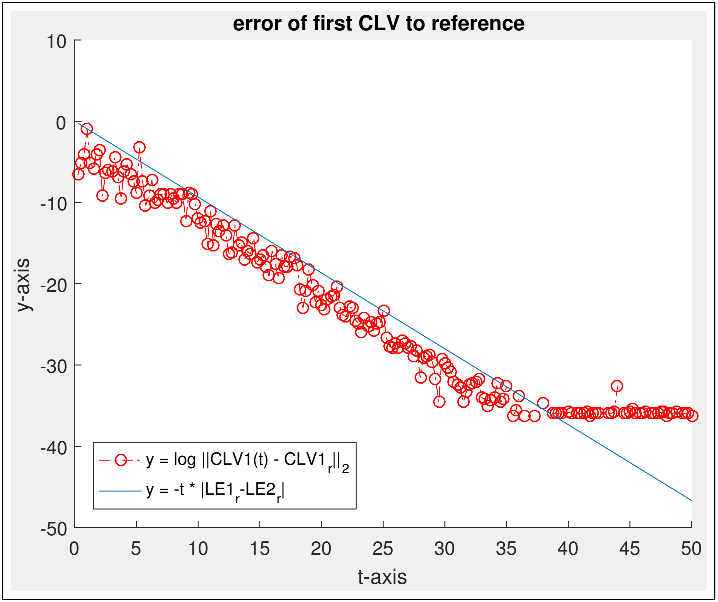
Convergence theorem in infinite dim.

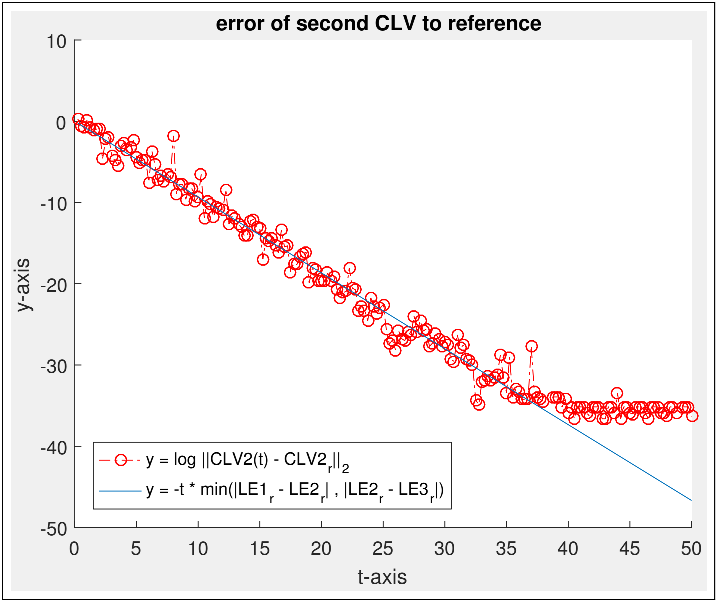
Lorenz 63 model

$$\dot{x} = \sigma(y - x)$$
$$\dot{y} = x(\rho - z) - y$$
$$\dot{z} = xy - \beta z$$

- classical parameters $\sigma = 10$, $\beta = 8/3$, $\rho = 28$
- simplified model for cellular convection
- known for its chaotic behavior (*butterfly effect*)
- Lorenz attractor
- SRB-measure and ergodicity


data trajectory
z-axis
50
40
30
20
10
0
-20
10
20
x-axis
-40
-20
0
20
40
y-axis
past data
future data
current state


error of first CLV to reference
10
0
-10
-20
-30
-40
-50
y-axis
0
5
10
15
20
25
30
35
40
45
50
t-axis
y = log ||CLV1(t) - CLV1r||2
y = -t * |LE1r-LE2r|


error of second CLV to reference
y-axis
t-axis
10
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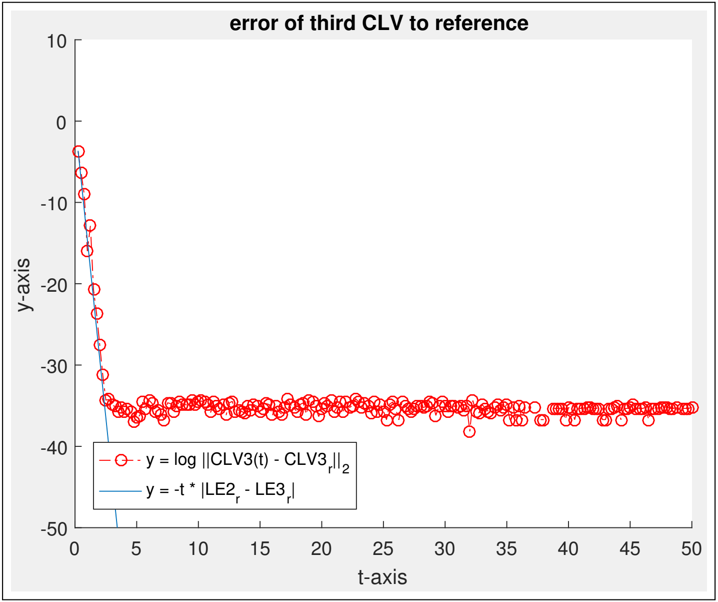
-10
-20
-30
-40
-50
0
5
10
15
20
25
30
35
40
45
50
y = log ||CLV2(t) - CLV2_r||_2
y = -t * min(|LE1_r - LE2_r| , |LE2_r - LE3_r|)


error of third CLV to reference
y-axis
t-axis
10
0
-10
-20
-30
-40
-50
0
5
10
15
20
25
30
35
40
45
50
y = log ||CLV3(t) - CLV3_r||_2
y = -t * |LE2_r - LE3_r|

Convergence theorems in finite dim.[13]

Theorem (N. 2019)

Assuming the MET on $\mathbb{R}^d$ with *discrete time*.

Ginelli's algorithm converges for *almost every* initial condition. The convergence is exponentially fast with a rate given by the spectral gap between associated Lyapunov exponents.

$$\limsup_{t\to\infty} \frac{1}{t} \log d(\mathrm{Approx}_i(t), \mathrm{CLV}_i) \leq -\min(|\lambda_i - \lambda_{i-1}|, |\lambda_i - \lambda_{i+1}|)$$

[13] Florian Noethen. "A projector-based convergence proof of the Ginelli algorithm for covariant Lyapunov vectors". In: *Physica D: Nonlinear Phenomena* 396 (2019), pp. 18–34. ISSN: 01672789. DOI: 10.1016/j.physd.2019.02.012.

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Theorem (N. 2019)

The above theorem holds for continuous time with convergence in measure instead of convergence almost everywhere.

[13] Florian Noethen. "A projector-based convergence proof of the Ginelli algorithm for covariant Lyapunov vectors". In: *Physica D: Nonlinear Phenomena* 396 (2019), pp. 18–34. ISSN: 01672789. DOI: 10.1016/j.physd.2019.02.012.

Convergence theorem in infinite dim.[14]

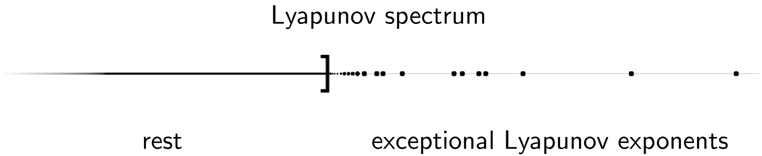
Theorem (N. 2019)

Assuming the MET on separable Hilbert spaces with discrete time.

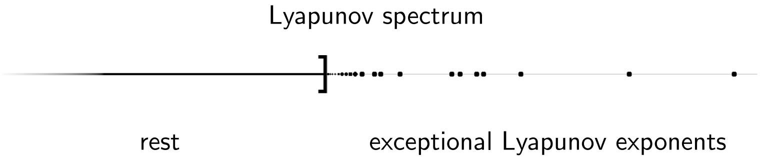
Ginelli's algorithm converges for almost every initial condition. The convergence is exponentially fast with a rate given by the spectral gap between associated Lyapunov exponents.

[14] Florian Noethen. "Computing covariant Lyapunov vectors in Hilbert spaces". In: (2019). arXiv: 1907.12458v1. URL: http://arxiv.org/pdf/1907.12458v1.

Differences to finite dim.



Differences to finite dim.



- CLVs only for highest Lyapunov exponents

Differences to finite dim.

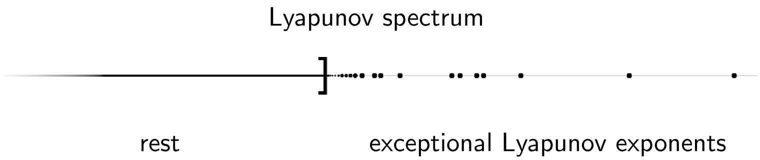
Lyapunov spectrum

rest

exceptional Lyapunov exponents

- CLVs only for highest Lyapunov exponents
- possibly non-invertible linear propagator

Differences to finite dim.



- CLVs only for highest Lyapunov exponents
- possibly non-invertible linear propagator
- no equivalent to Lebesgue measure
- almost everywhere in the sense of prevalence[15]

[15] William Ott and James A. Yorke. "Prevalence". In: *Bulletin of the American Mathematical Society* 42.3 (2005), pp. 263–290. ISSN: 0273-0979. DOI: 10.1090/S0273-0979-05-01060-8.

Conclusions

- classical stability theory $\xrightarrow{\text{MET}}$ general trajectories
- Ginelli's algorithm to compute CLVs

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- Ginelli's algorithm to compute CLVs

⇒ **convergence theorems in finite dim.**

- discrete time: convergence a.e. (exp. fast)
- continuous time: convergence in measure (exp. fast)

Conclusions

- classical stability theory $\xrightarrow{\text{MET}}$ general trajectories
- Ginelli's algorithm to compute CLVs

$\Rightarrow$ **convergence theorems in finite dim.**

- discrete time: convergence a.e. (exp. fast)
- continuous time: convergence in measure (exp. fast)

$\Rightarrow$ **convergence theorem in infinite dim.**

- semi-invertible setting
- convergence a.e. (exp. fast)