An energy conserving, upwinded compatible finite element discretisation for the dry compressible Euler equations

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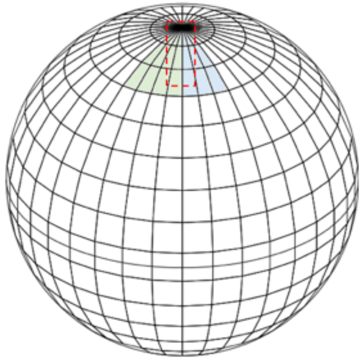
Outline

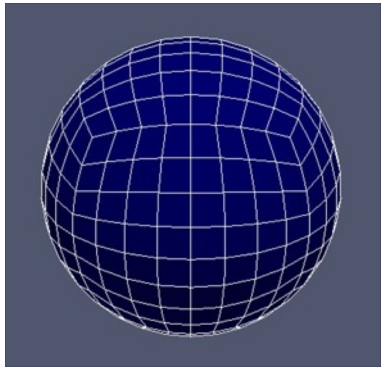
1. Background
   - Compatible finite elements
   - Hamiltonian framework
   - Upwind stabilization methods

2. Hamiltonian framework including upwinding for cFEM
   - Results for compressible Euler equations
   - Computational efficiency
   - Interpretation

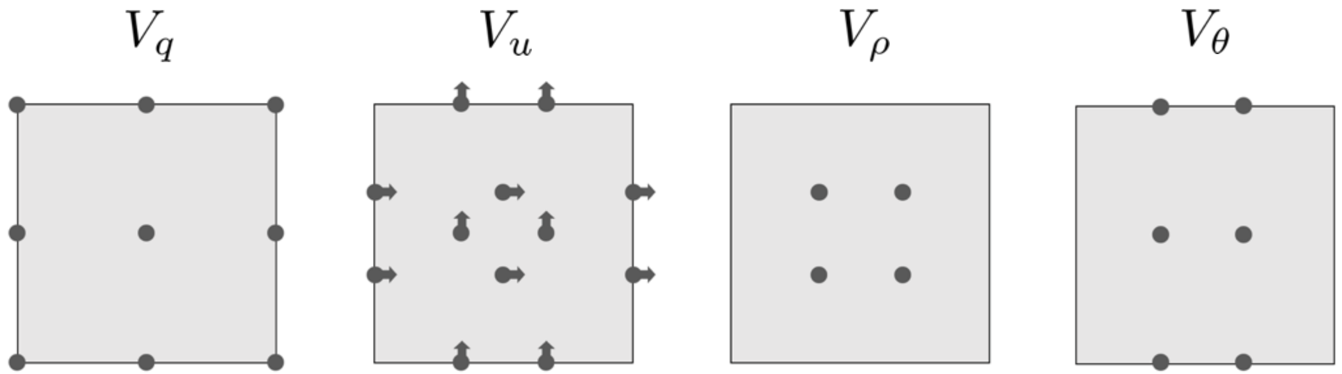
Compatible finite elements

- Used in UK Met Office’s next dynamical core GungHo
  - FEM extension of Arakawa C Grid
  - Allows for general grids
- Map FE spaces via differential operators





$$
\begin{array}{ccccc}
H^1(\Omega) & \xrightarrow{\nabla\times} & H(\mathrm{div};\Omega) & \xrightarrow{\nabla\cdot} & L^2(\Omega) \\
\downarrow \pi^0 & & \downarrow \pi^1 & & \downarrow \pi^2 \\
V^0(\Omega) & \xrightarrow{\nabla\times} & V^1(\Omega) & \xrightarrow{\nabla\cdot} & V^2(\Omega)
\end{array}
$$



Vertical slice elements

CP staggering

Energy conservation in NWP

$$\frac{D\mathbf{u}}{Dt} + 2\boldsymbol{\Omega} \times \mathbf{u} + c_p \theta \nabla \pi = -g\mathbf{k}$$

$$\frac{D\rho}{Dt} + \rho \nabla \cdot \mathbf{u} = 0$$

$$\frac{D\theta}{Dt} = 0$$

$$E = K + P + I$$

$$\frac{dE}{dt} = \frac{dK}{dt} + \frac{dP}{dt} + \frac{dI}{dt} = 0$$

- Failure to account for a numerically closed energy budget may lead to net energy biases in climate models
  - Partly due to missing reinjection $K \rightarrow I$

Energy conservation in NWP

$$\frac{D\mathbf{u}}{Dt} + 2\mathbf{\Omega} \times \mathbf{u} + c_p\theta\nabla\pi = -g\mathbf{k}$$

$$\frac{D\rho}{Dt} + \rho\nabla\cdot\mathbf{u} = 0$$

$$\frac{D\theta}{Dt} = 0$$

$$E = K + P + I$$

$$\frac{dE}{dt} = \frac{dK}{dt} + \frac{dP}{dt} + \frac{dI}{dt} = 0$$

$$\frac{dE}{dt} = \frac{\delta E}{\delta \mathbf{u}}\frac{d\mathbf{u}}{dt} + \frac{\delta E}{\delta \rho}\frac{d\rho}{dt} + \frac{\delta E}{\delta \theta}\frac{d\theta}{dt} \xrightarrow{?}$$ Discretized equations

Hamiltonian framework

- Hamiltonian representing total energy $$H(\mathbf{u},\rho,\theta) = \int_\Omega (\frac{\rho}{2}|\mathbf{u}|^2 + g\rho z + c_v \rho\theta\pi)dx$$

- Poisson bracket
  - Bilinear form
  - Antisymmetric

$$\{F,H\} = -\langle\frac{\delta F}{\delta \mathbf{u}}, \omega \times \frac{\delta H}{\delta \mathbf{u}}\rangle + \langle\nabla \cdot \frac{\delta F}{\delta \mathbf{u}}, \frac{\delta H}{\delta \rho}\rangle + \langle\frac{\delta H}{\delta \theta}, \frac{1}{\rho}\frac{\delta F}{\delta \mathbf{u}} \cdot \nabla\theta\rangle$$
$$-\langle\nabla \cdot \frac{\delta H}{\delta \mathbf{u}}, \frac{\delta F}{\delta \rho}\rangle - \langle\frac{\delta F}{\delta \theta}, \frac{1}{\rho}\frac{\delta H}{\delta \mathbf{u}} \cdot \nabla\theta\rangle$$

$$\frac{\delta H}{\delta \mathbf{u}} = \rho\mathbf{u} \qquad \frac{\delta H}{\delta \rho} = \frac{1}{2}|\mathbf{u}|^2 + gz + c_p\theta\pi \qquad \frac{\delta H}{\delta \theta} = c_p\rho\pi$$

$$\pi = (\frac{R}{p_0}\rho\theta)^{\frac{1-\kappa}{\kappa}}$$
$$\omega = \frac{1}{\rho}(\nabla \times \mathbf{u} + \mathbf{\Omega})$$

Hamiltonian framework

- Poisson system $$\frac{dF}{dt} = \{F, H\}$$

- Poisson bracket $$\{F,H\} = -\langle \frac{\delta F}{\delta \mathbf{u}}, \omega \times \frac{\delta H}{\delta \mathbf{u}} \rangle + \langle \nabla \cdot \frac{\delta F}{\delta \mathbf{u}}, \frac{\delta H}{\delta \rho} \rangle + \langle \frac{\delta H}{\delta \theta}, \frac{1}{\rho} \frac{\delta F}{\delta \mathbf{u}} \cdot \nabla \theta \rangle$$

$$-\langle \nabla \cdot \frac{\delta H}{\delta \mathbf{u}}, \frac{\delta F}{\delta \rho} \rangle - \langle \frac{\delta F}{\delta \theta}, \frac{1}{\rho} \frac{\delta H}{\delta \mathbf{u}} \cdot \nabla \theta \rangle$$

$$F = \langle \phi, \rho \rangle \quad \Rightarrow \quad \frac{\delta F}{\delta \rho} = \phi, \quad \langle \phi, \frac{\partial \rho}{\partial t} \rangle = -\langle \phi, \nabla \cdot (\rho \mathbf{u}) \rangle$$

$$F = \langle \gamma, \theta \rangle \quad \Rightarrow \quad \frac{\delta F}{\delta \theta} = \gamma, \quad \langle \gamma, \frac{\partial \theta}{\partial t} \rangle = -\langle \gamma, \mathbf{u} \cdot \nabla \theta \rangle$$

Hamiltonian framework

- Poisson system $$\frac{dF}{dt} = \{F, H\}$$

- Poisson bracket $$\{F,H\} = -\langle \frac{\delta F}{\delta \mathbf{u}}, \omega \times \frac{\delta H}{\delta \mathbf{u}} \rangle + \langle \nabla \cdot \frac{\delta F}{\delta \mathbf{u}}, \frac{\delta H}{\delta \rho} \rangle + \langle \frac{\delta H}{\delta \theta}, \frac{1}{\rho} \frac{\delta F}{\delta \mathbf{u}} \cdot \nabla \theta \rangle$$
$$- \langle \nabla \cdot \frac{\delta H}{\delta \mathbf{u}}, \frac{\delta F}{\delta \rho} \rangle - \langle \frac{\delta F}{\delta \theta}, \frac{1}{\rho} \frac{\delta H}{\delta \mathbf{u}} \cdot \nabla \theta \rangle$$

$$F = H \quad \Rightarrow \quad \frac{dH}{dt} = \{H, H\} = -\{H, H\} = 0$$

Poisson bracket – no upwinding

$$\frac{dF}{dt} = \{F, H\} \qquad \{F,H\} = -\langle \frac{\delta F}{\delta \mathbf{u}}, \omega \times \frac{\delta H}{\delta \mathbf{u}} \rangle + \langle \nabla \cdot \frac{\delta F}{\delta \mathbf{u}}, \frac{\delta H}{\delta \rho} \rangle + \langle \frac{\delta H}{\delta \theta}, \frac{1}{\rho} \frac{\delta F}{\delta \mathbf{u}} \cdot \nabla \theta \rangle$$

$$-\langle \nabla \cdot \frac{\delta H}{\delta \mathbf{u}}, \frac{\delta F}{\delta \rho} \rangle - \langle \frac{\delta F}{\delta \theta}, \frac{1}{\rho} \frac{\delta H}{\delta \mathbf{u}} \cdot \nabla \theta \rangle$$

$$\frac{\delta H}{\delta \mathbf{u}} = \mathbf{F} = P_{V_\mathbf{u}}(\rho \mathbf{u})$$

$$\frac{\delta H}{\delta \rho} = P = P_{V_\rho}(\frac{1}{2}|\mathbf{u}|^2 + gz + c_p \theta \pi)$$

$$\frac{\delta H}{\delta \theta} = T = P_{V_\theta}(c_p \rho \pi)$$

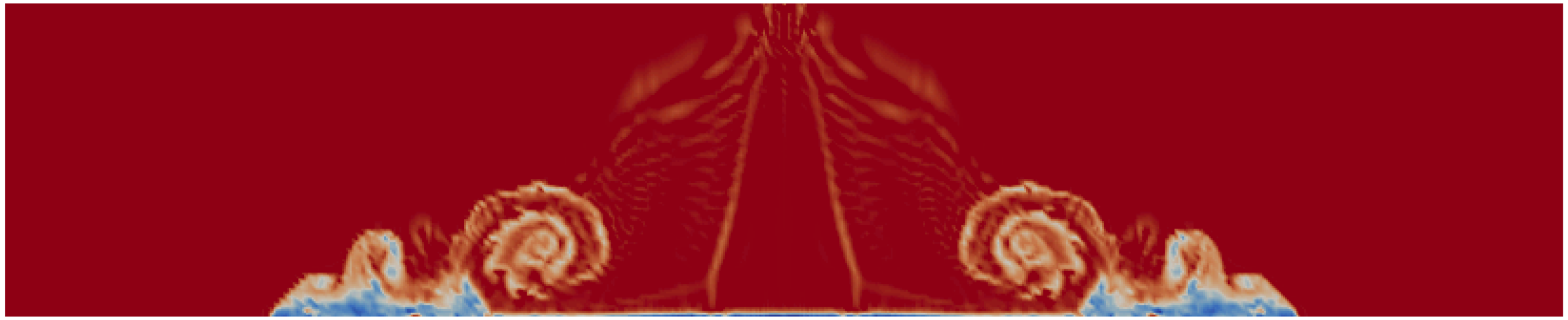
$$\langle \mathbf{w}, \mathbf{u}_t + \omega \times \mathbf{F} \rangle - \langle \nabla \cdot \mathbf{w}, P \rangle - \langle \mathbf{w}, T/\rho \nabla \theta \rangle = 0$$

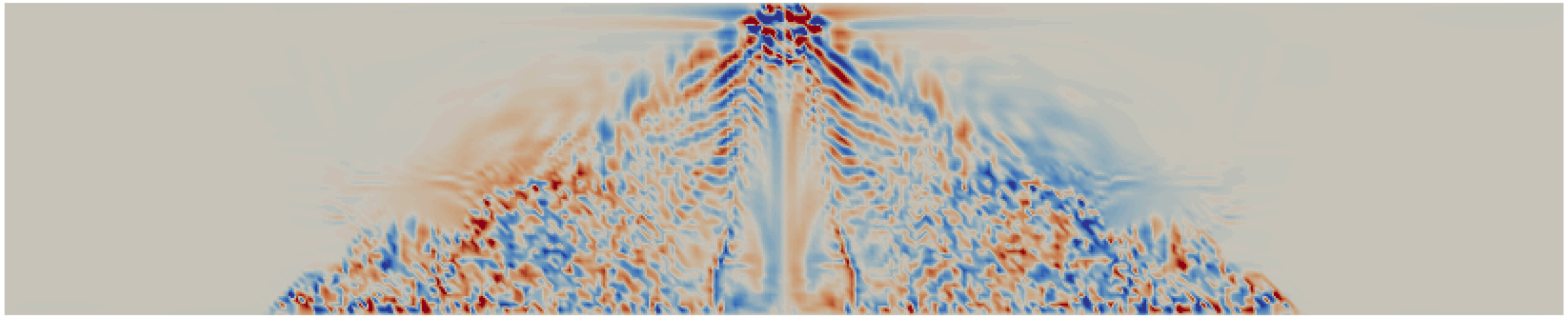
$$\langle \phi, \rho_t + \nabla \cdot \mathbf{F} \rangle = 0$$

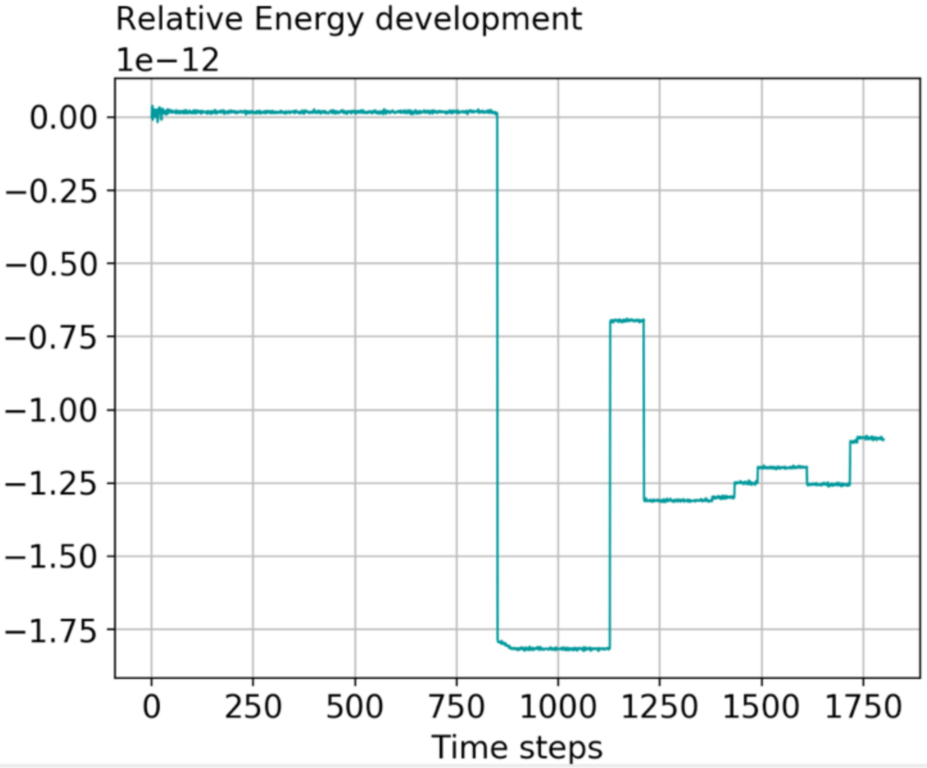
$$\langle \gamma, \theta_t + \mathbf{F}/\rho \cdot \nabla \theta \rangle = 0$$

Test results – dry compressible Euler

- Falling cold bubble test case







Test results – dry compressible Euler

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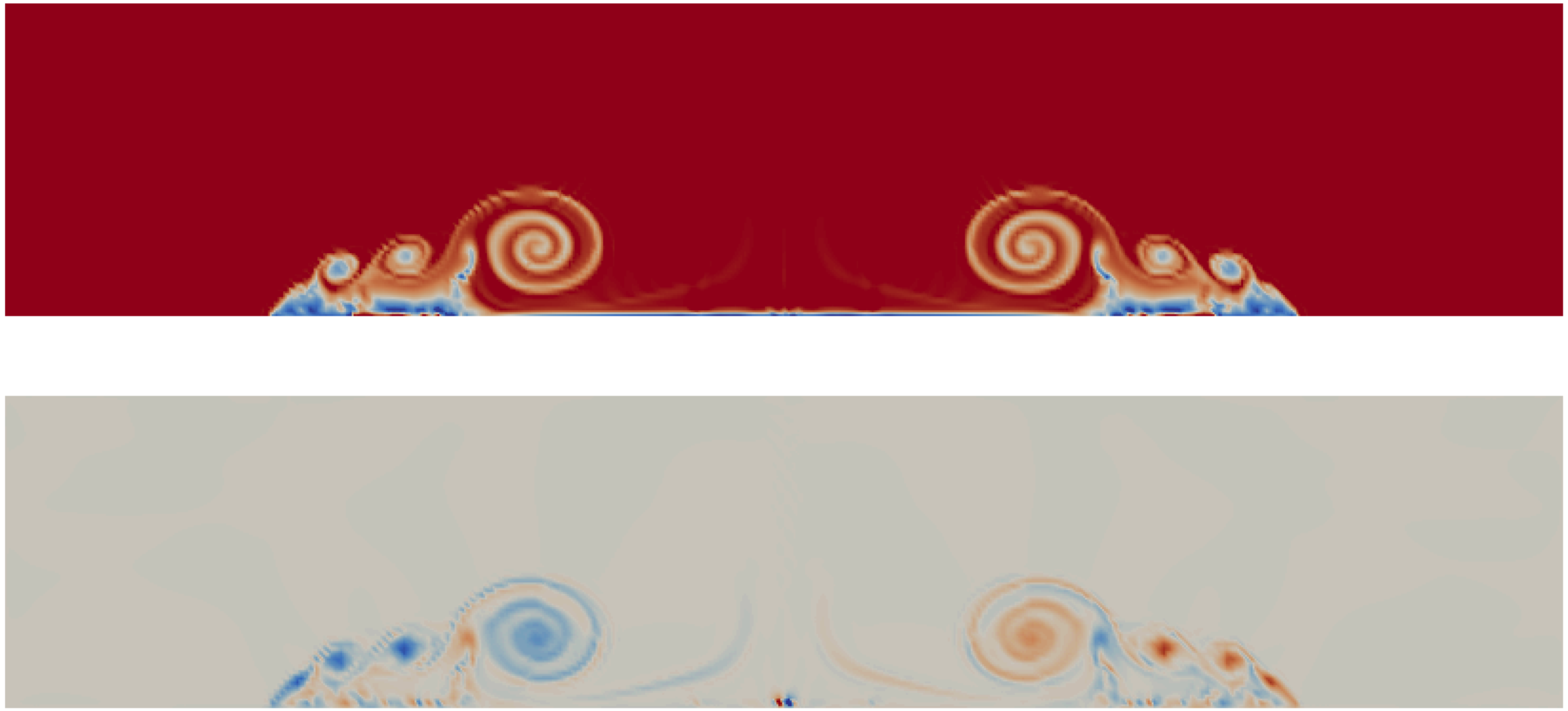
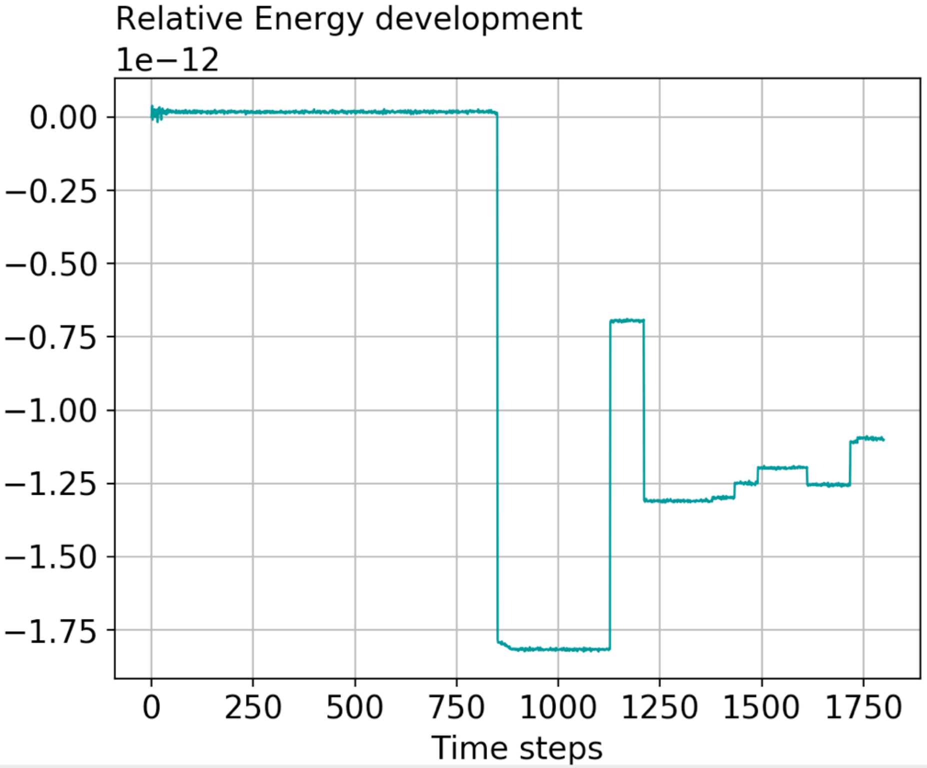


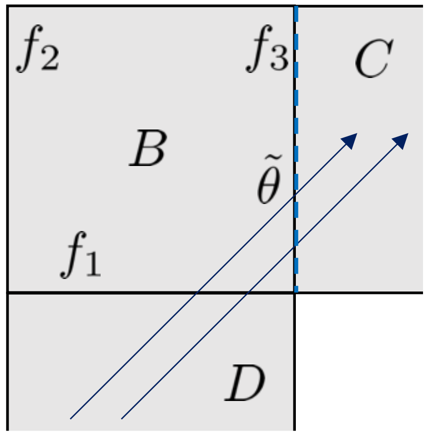


Upwind stabilization

- Different finite element spaces require different advection schemes
  - Horizontal DG upwinding for potential temperature θ

$$\langle \gamma, \mathbf{u}\cdot\nabla\theta\rangle \longrightarrow -\langle \nabla\cdot(\gamma\mathbf{u}), \theta\rangle + \int_\Gamma [\![\gamma\mathbf{u}]\!]\tilde{\theta}\, dS$$

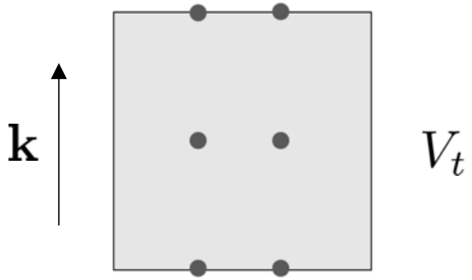
$$\longrightarrow \langle \gamma, \mathbf{u}\cdot\nabla\theta\rangle + \int_\Gamma \left([\![\gamma\mathbf{u}]\!]\tilde{\theta} - [\![\gamma\mathbf{u}\theta]\!]\right) dS = L_{\mathbf{u}}(\theta;\gamma)$$



  - Vertical SUPG

$$\gamma \longrightarrow \gamma + \tau(\mathbf{k}\cdot\mathbf{u})(\mathbf{k}\cdot\nabla\gamma) = \gamma + \tilde{\gamma}_{\mathbf{u}}$$

$$\langle \gamma + \tilde{\gamma}_{\mathbf{u}}, \theta_t\rangle = L_{\mathbf{u}}(\theta, \gamma + \tilde{\gamma}_{\mathbf{u}})$$



Potential temperature

- Move upwind part $\tilde{\gamma}_{\mathbf{u}}$ into bracket

$$\langle \gamma + \tilde{\gamma}_{\mathbf{u}}, \theta_t \rangle = L_{\mathbf{u}}(\theta, \gamma + \tilde{\gamma}_{\mathbf{u}}) \longrightarrow \langle \gamma, \theta_t \rangle = L_{\mathbf{u}}(\theta, \gamma + \tilde{\gamma}_{\mathbf{u}}) - \langle \tilde{\gamma}_{\mathbf{u}}, \theta_t \rangle$$

$$\frac{dF}{dt} = \{F, H\} \quad \text{for} \quad F = \langle \gamma, \theta \rangle$$

$$+\langle \frac{\delta H}{\delta \theta}, \frac{1}{\rho}\frac{\delta F}{\delta \mathbf{u}} \cdot \nabla \theta \rangle \qquad + L^{\theta}_{\mathbb{U}(\rho, \frac{\delta F}{\delta \mathbf{u}})}\Big(\theta; \frac{\delta H}{\delta \theta} + \Big(\widetilde{\frac{\delta H}{\delta \theta}}\Big)_{\mathbf{u}}\Big) + \langle \Big(\widetilde{\frac{\delta H}{\delta \theta}}\Big)_{\mathbf{u}}, \theta_t \rangle$$

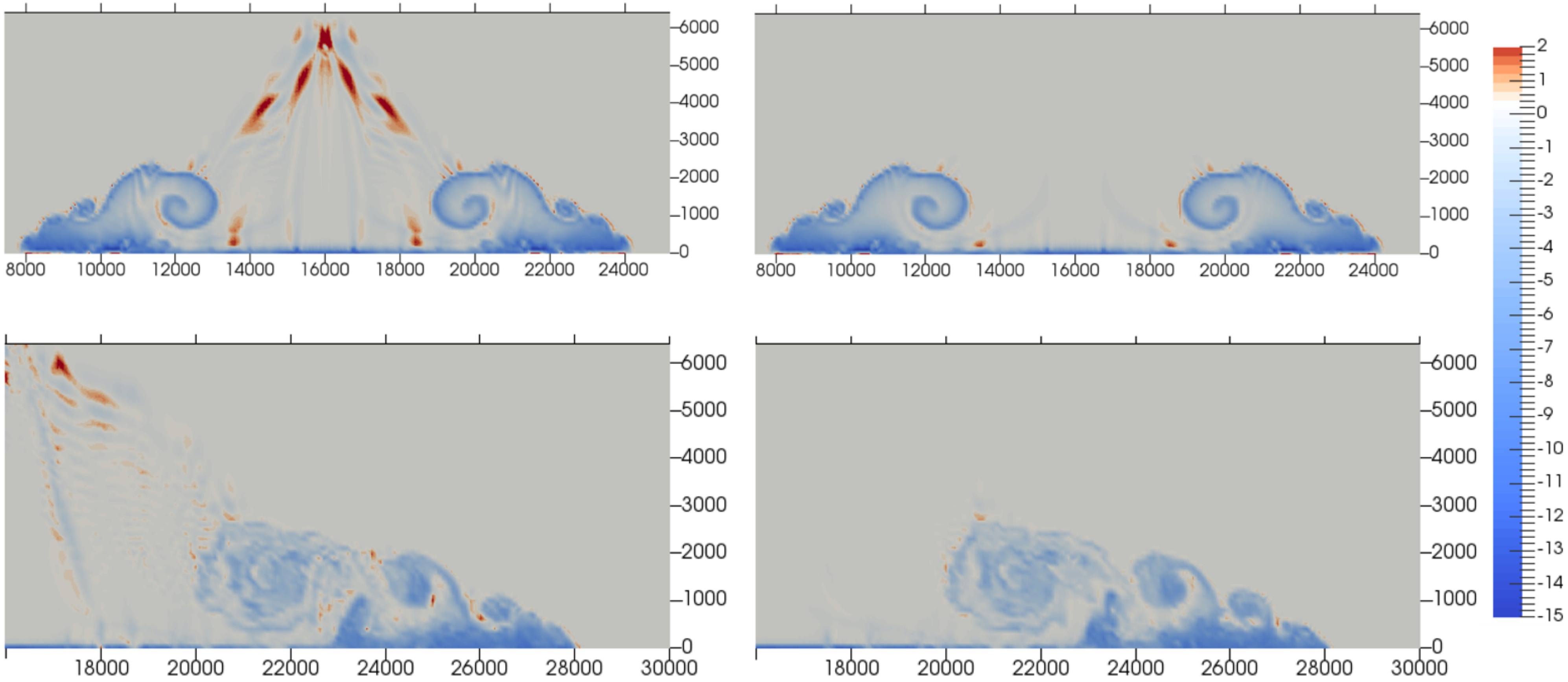
$$-\langle \frac{\delta F}{\delta \theta}, \frac{1}{\rho}\frac{\delta H}{\delta \mathbf{u}} \cdot \nabla \theta \rangle \qquad - L^{\theta}_{\mathbb{U}(\rho, \frac{\delta H}{\delta \mathbf{u}})}\Big(\theta; \frac{\delta F}{\delta \theta} + \Big(\widetilde{\frac{\delta F}{\delta \theta}}\Big)_{\mathbf{u}}\Big) - \langle \Big(\widetilde{\frac{\delta F}{\delta \theta}}\Big)_{\mathbf{u}}, \theta_t \rangle$$

DG/SUPG →

Results

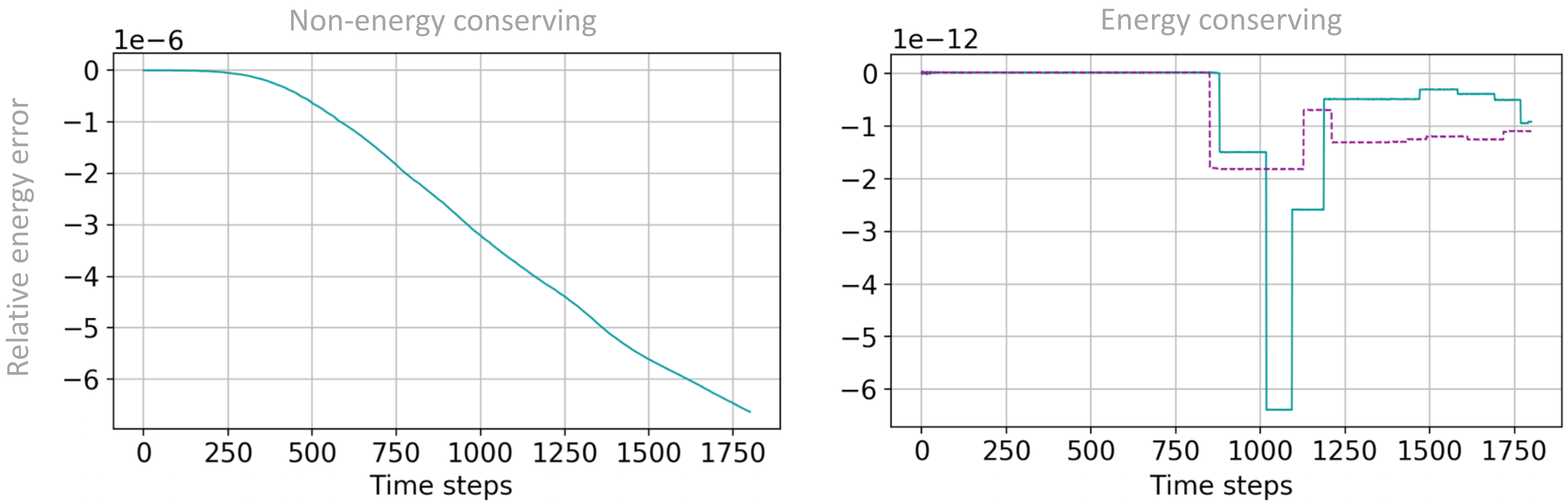
- Falling cold air bubble

Potential temperature



Results

- Falling cold air bubble



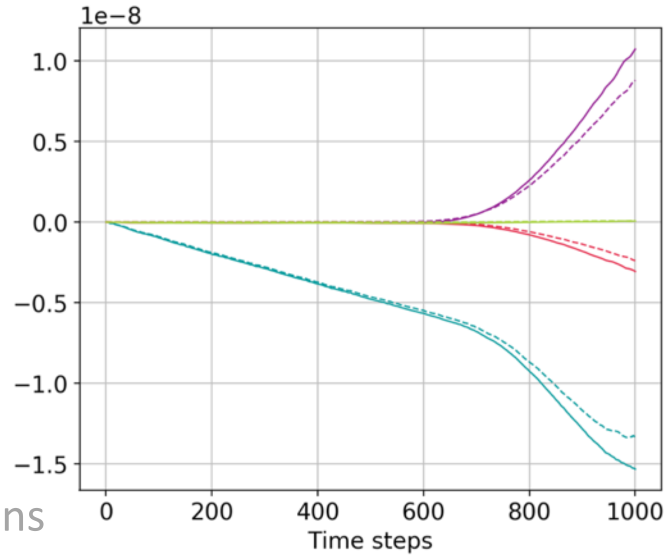
Computational cost

- Revert to approximately energy conserving time discretization

$$\overline{\frac{\delta H}{\delta \mathbf{u}}} = \frac{1}{3} P_{V_u} \left(\rho^n \mathbf{u}^n + \frac{1}{2} \rho^{n+1} \mathbf{u}^n + \frac{1}{2} \rho^n \mathbf{u}^{n+1} + \rho^{n+1} \mathbf{u}^{n+1} \right) \quad \longrightarrow \quad \overline{\frac{\delta H}{\delta \mathbf{u}}}' = P_{V_u} (\bar{\rho} \bar{\mathbf{u}})$$

  - Can rearrange Poisson bracket swapping upwind and advecting velocities
  - Remaining projections:

$$\overline{\frac{\delta H}{\delta \rho}} \in V_\rho \qquad \overline{\frac{\delta H}{\delta \theta}} \in V_\theta$$

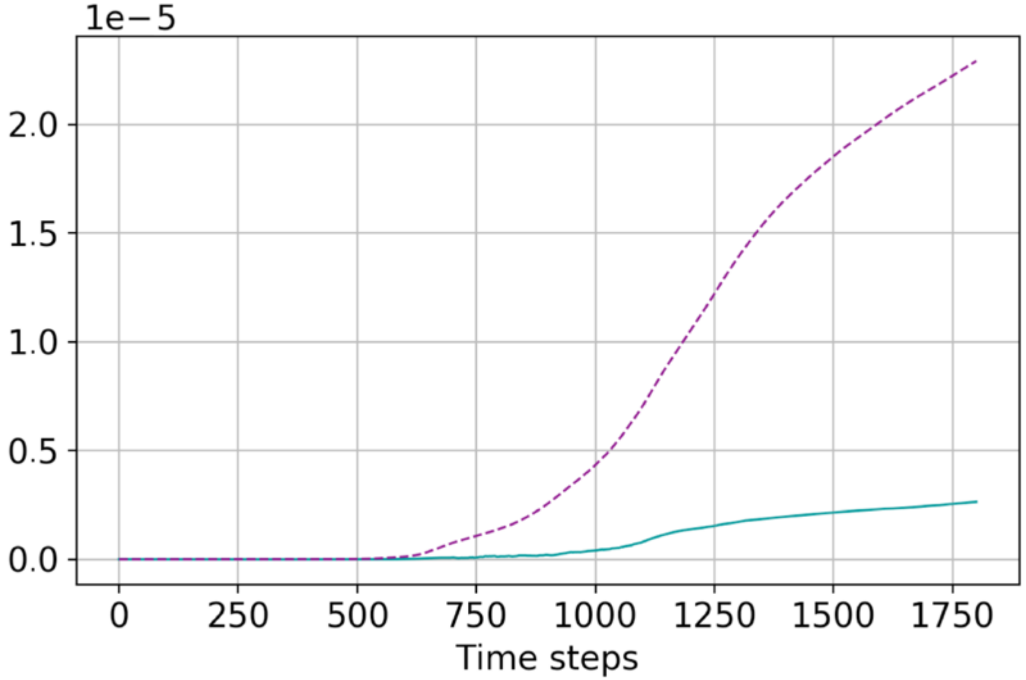


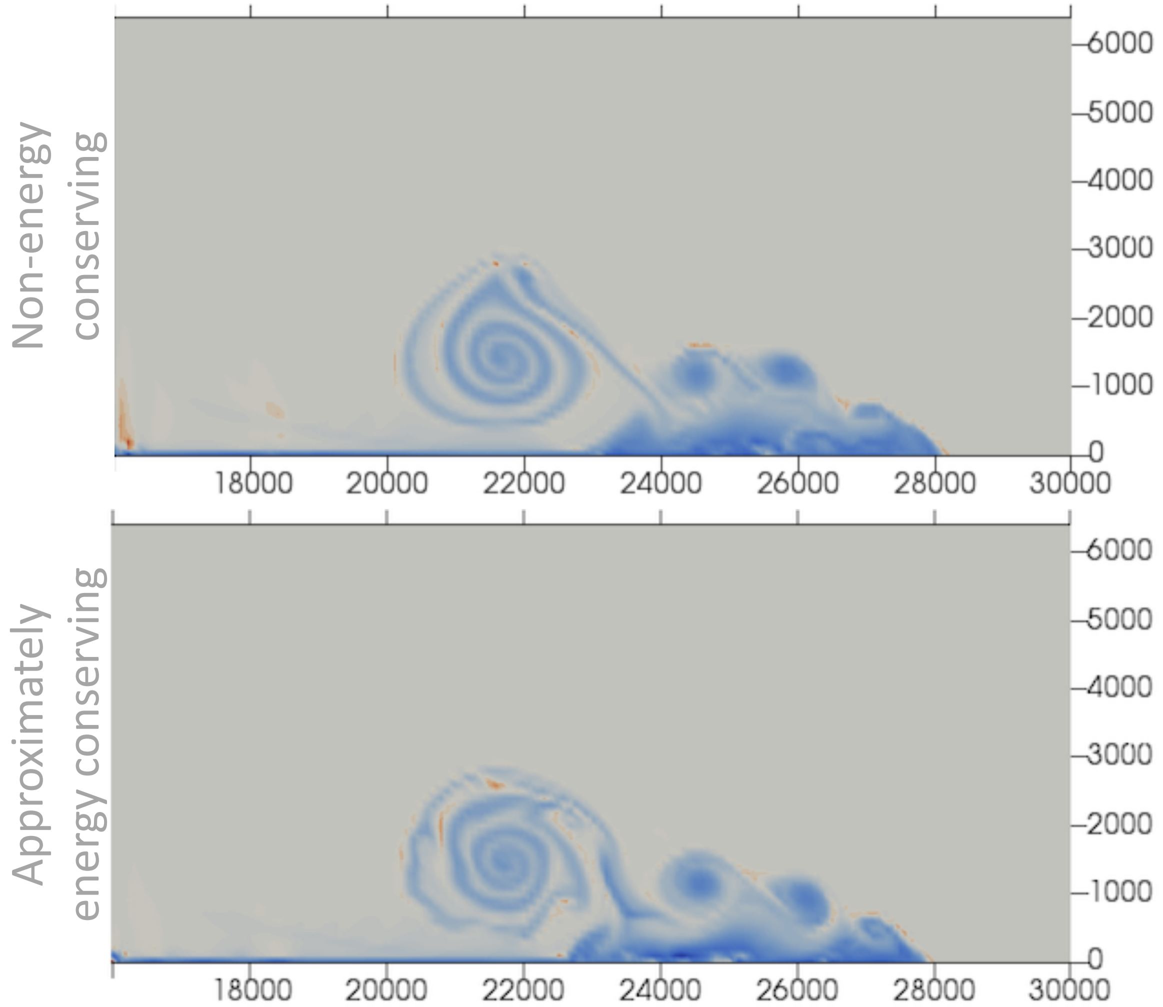
Relative energy error relative to number of quasi-Newton iterations

Results

- Falling cold air bubble

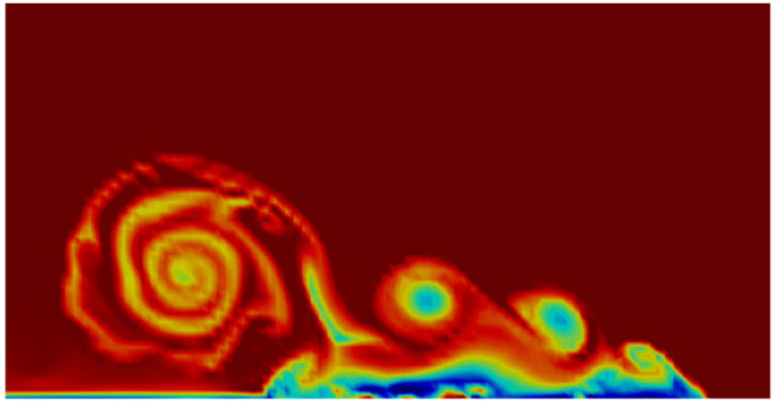
Relative energy error, k=4



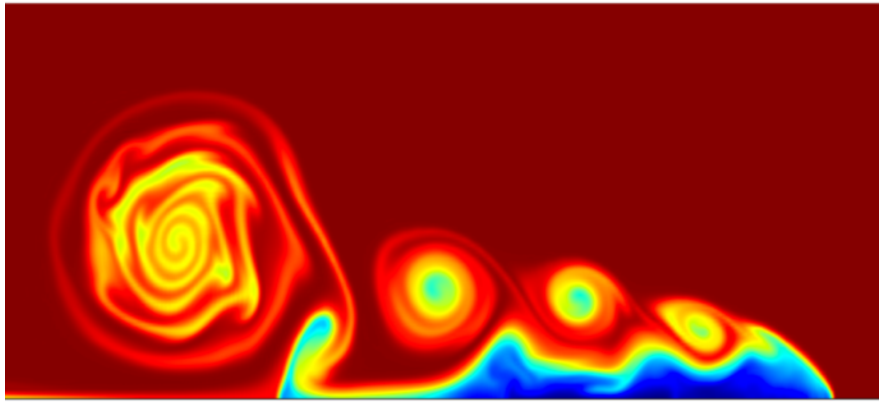


Results

Approximately energy conserving



Dynamic sub-grid scale model



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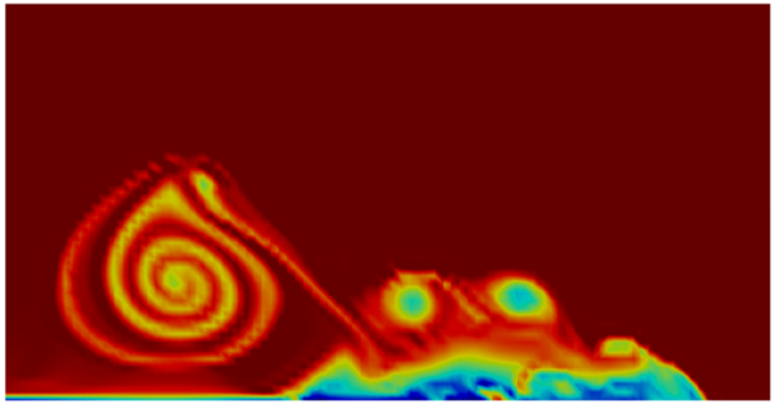
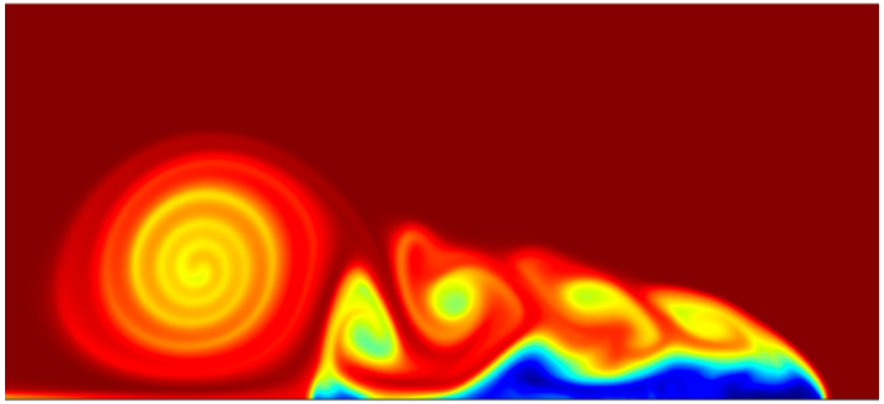
Non-energy conserving



Smagorinsky sub grid scale model



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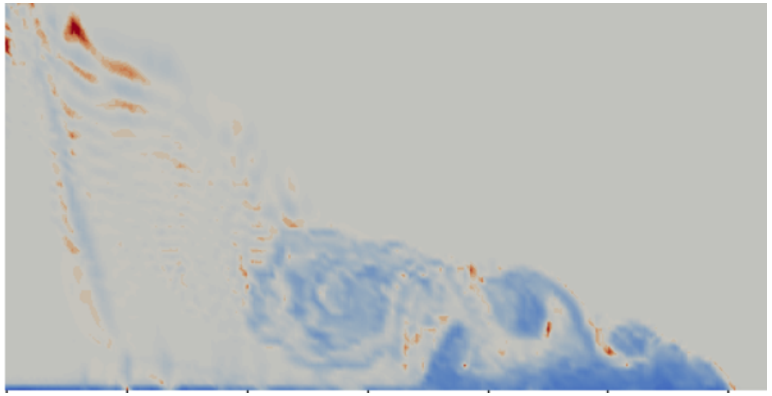
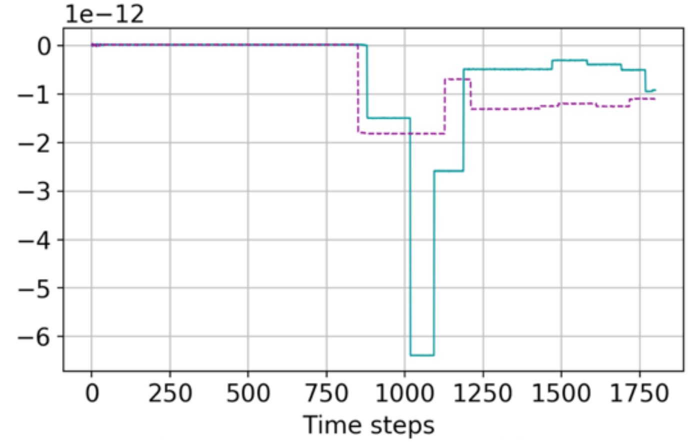
Summary

- Fully upwinded, energy conserving formulation





- Simplification to reduce computational cost

$$\overline{\frac{\delta H}{\delta \mathbf{u}}} \longrightarrow \overline{\frac{\delta H}{\delta \mathbf{u}}}'$$

$$\mathbb{U}\left(\bar{\rho}, \overline{\frac{\delta H}{\delta \mathbf{u}}}\right), \quad (\mathbb{S}_{\mathbf{u}}^{-1})^*, \quad (\theta^a)^* \in V_u, \quad \overline{\frac{\delta H}{\delta \rho}} \in V_\rho, \quad \overline{\frac{\delta H}{\delta \theta}} \in V_\theta$$

References

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- G.A. Wimmer, C.J. Cotter, W. Bauer. Energy conserving SUPG methods for compatible finite element schemes in numerical weather prediction, ArXiv preprint 2001.09590 (2020)
- S. Marras, M. Nazarov, F.X. Giraldo. Stabilized higher-order Galerkin methods based on a parameter-free dynamic SGS model for LES, Journal of Computational Physics, 301: 77-101 (2015)