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Generalized LSG models with spatially varying Coriolis parameter

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In this paper, we derive and study approximate balance models for nearly geostrophic shallow water flow where the Coriolis parameter is permitted to vary across the domain so long as it remains nondegenerate. This situation includes, for example, the β-plane approximation to the shallow water equations at mid-latitudes. Our approach is based on changing configuration space coordinates in the underlying variational principle in such a way that that consistent asymptotics in the transformed Lagrangian leads to a degenerate Lagrangian structure. In this article, we restrict our attention to first order models. We show that the resulting models can be formulated in terms of an advected potential vorticity with a nonlinear vorticity inversion relation. We study the associated solvability conditions and identify a subfamily of models for which these conditions are satisfied without additional restrictions on the data. Finally, we provide the link between our framework and the theory of constrained Hamiltonian systems.

Keywords: Balance models, varying Coriolis parameter, variational asymptotics

1. Introduction

Large-scale flow in mid-latitude atmosphere and ocean dynamics is characterized by smallness of the Rossby number, which measures the relative importance of inertial vs. Coriolis forces. To leading order in Rossby number, such flow is in geostrophic balance—the pressure gradient balances the Coriolis force exactly, and the flow is stationary. A balance model then describes the slow dynamics of small departures from a balanced state. In the simplest case, when the full flow is described by the rotating shallow water equations as we assume throughout this paper, there are two classical balance models, the semigeostrophic and the quasigeostrophic equations (see, e.g., the text book expositions of Pedlosky 1987 and Salmon 1998), which differ in the assumed scaling of a second parameter, the Burger number, and in the scaling of the surface height variations. In this paper, we shall only be concerned with the semigeostrophic limit where Burger and Rossby numbers are of the same order and there are no restrictions on the magnitude of surface height variations except for a natural positivity condition on the layer depth.

Salmon (1985) pioneered the derivation of balance models via the variational formulation of the fluid system and introduced two new models, the so-called L_1 model and the large-scale semigeostrophic (LSG) equations. (The term "large-scale semigeostrophic equations" was coined in Salmon (1996), where the author implements similar ideas for a stratified flow.) His ideas were subsequently extended in a number of ways (see Shutts 1989, Holm 1996, Purser 1999, Wunderer 2001, McIntyre and Roulstone 2002, Vanneste and Bokhove 2002, Oliver 2006,

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and references therein). In this paper, we revisit the approach of Oliver (2006), who introduced a general framework based on writing the variational principle in a new coordinate system which is chosen precisely so that, when consistently truncated to a certain order in the Rossby number, the variational structure degenerates, thereby providing an implicit constraint on the dynamics.

When the Coriolis parameter is constant, this approach yields a one-parameter family of balance models, the *generalized LSG equations*. As a function of the model parameter, they "interpolate" between Salmon's L_1 model and the LSG equations. For a fixed value of the model parameter, an instance of the generalized LSG equations is, in many respects, similar to Hoskins' semigeostrophic equations. Both sets of equations are Hamiltonian (for the semigeostrophic equations, see Salmon 1985, for the generalized LSG equations, see Oliver and Vasylkevych 2011), both coincide up to terms of order one in Rossby number, and, in the case of constant Coriolis parameter, both can be formulated as an advection equation for the potential vorticity in a transformed coordinate system coupled with a nonlinear potential vorticity inversion. In the semigeostrophic case, the transformation has been introduced by Hoskins (1975) and is now known by his name; the associated potential vorticity inversion law is a nonlinear elliptic Monge–Ampère equation. Generalized LSG theory also employs separate computational coordinate system, in which advected potential vorticity is coupled to the velocity by the system of elliptic PDEs. The key difference is that the Hoskins transformation into semigeostrophic coordinates is explicit in the physical coordinate system and implicit in the new semigeostrophic coordinates. For the generalized LSG equations, the situation is reversed, which has an obvious benefit for the numerical implementation of the model. Advection of potential vorticity was used to prove well-posedness for the semigeostrophic equations (Benamou and Brenier 1998) and for the generalized LSG equations (Çalik, Oliver and Vasylkevych 2012). Finally, for a constant Coriolis parameter, the semigeostrophic equations also possess a materially conserved potential vorticity in physical coordinates. We remark that there are other classes of balance models, e.g. the quasigeostrophic equations and higher order extensions, which can also be formulated in terms of potential vorticity advection and inversion (see Vallis 2006, and references therein).

When the Coriolis parameter is spatially varying, there is no known conserved potential vorticity for the semigeostrophic equations in physical coordinates (Roulstone and Sewell 1996). A conserved potential vorticity does exist in so-called vorticity coordinates (Schubert and Magnusdottir 1994, Roulstone and Sewell 1996, 1997), but computing the transformation to vorticity coordinates requires another prognostic equation (Schubert and Magnusdottir 1994). More recently, Cullen *et al.* (2005) use the theory of optimal transport to give a formal argument that the semigeostrophic equations on a sphere can be written in terms of potential vorticity advection and inversion, but in order to obtain a practical solution procedure, they continue to work in physical coordinates. Moreover, to our knowledge there are no known results on the mathematical well-posedness of the semigeostrophic equations in this general case.

In this paper, we extend the strategy of Oliver (2006) to the case of the rotating shallow water equations with spatially varying Coriolis parameter. We assume that the Coriolis parameter f is a smooth function and that it remains bounded away from zero; however, no further restrictions are made. In our setting, the difficulties to semigeostrophic theory posed by spatial variations of f largely disappear. We find that the equations of motion can be derived in much the same way as for nonvarying f, and that they can be formulated as an advection equation for a transformed potential vorticity (PV) coupled with a nonlinear potential vorticity inversion relation. The transformation back to physical coordinates is explicit in the new coordinates and can be readily computed.

Invertibility of the potential vorticity relation across the family of generalized LSG models is guaranteed if either the Rossby number or the gradient of the Coriolis parameter is sufficiently small. With certain particular choices of the parameters of the transformation (which, however, do not include a case analogous to Salmon's L_1 model for nonconstant f), invertibility hinges only on the positivity of the Coriolis parameter and of the initial potential vorticity. This condition is already necessary in the case of non-varying Coriolis parameter (Çalik, Oliver and Vasylkevych 2012) and appears to be both sharp and physically reasonable. It unconditionally includes the β-plane approximation to the shallow water equations at mid-latitudes. However, we cannot deal with the degeneracy of the Coriolis parameter for the spherical shallow water equations at the equator—a fundamental difficulty for balance models in general.

The paper is structured as follows. Section 2 introduces the shallow water equations, the semigeostrophic scaling, and the variational formulation. Section 3 recalls the results from Oliver (2006) on the degenerate variational setting for the balance models. In Section 4, we derive general first order balance models with non-constant Coriolis parameter via the transformational approach. Section 5 looks at the vorticity formulation of the resulting models and discusses solvability of the PV inversion. In section 6, we point out two distinct families of generalized LSG models and discuss the solvability of their PV inversion and their relation to the L_1 and the basic LSG model. Section 7 gives a brief reinterpretation of our method as a constrained Hamiltonian system in the spirit of Salmon. Short concluding remarks comprise the final section of this paper.

2. Shallow water equations and scaling

The parent model from which our balance models are derived and to which they must be compared is the system of rotating shallow water equations (Pedlosky 1987, Salmon 1998), which describe the vertically averaged motion of a shallow layer of an inviscid homogeneous fluid on a rotating plane. In non-dimensionalized form, they read

$$\varepsilon\,(\partial_t + \boldsymbol{u}\cdot\boldsymbol{\nabla})\boldsymbol{u} + f\,\boldsymbol{u}^\perp + \varepsilon^{-1}\,B\,\boldsymbol{\nabla}(h-b) = 0\,, \tag{1a}$$

$$\partial_t h + \boldsymbol{\nabla}\cdot(h\boldsymbol{u}) = 0\,, \tag{1b}$$

where $\boldsymbol{u} = \boldsymbol{u}(\boldsymbol{x},t)$ is the horizontal fluid velocity, $h = h(\boldsymbol{x},t)$ the total depth of the layer, $b = b(\boldsymbol{x}) \geq b_{\min} > 0$ the depth at equilibrium, and $f = f(\boldsymbol{x}) \geq f_{\min} > 0$ the non-dimensionalized Coriolis parameter. We write $\boldsymbol{u} = (u_1, u_2)$, $\boldsymbol{u}^\perp = (-u_2, u_1)$, $\boldsymbol{x} = (x_1, x_2)$, $\boldsymbol{\nabla} = (\partial_1, \partial_2)$, and $\boldsymbol{\nabla}^\perp = (-\partial_2, \partial_1)$.

The asymptotic regime is governed by two non-dimensional quantities, the Rossby number ε and the Burger number B, defined by

$$\varepsilon = \frac{U_0}{f_0\,L_0} \qquad \text{and} \qquad B = \frac{g\,H_0}{f_0^2\,L_0^2}\,, \tag{2}$$

where g is the constant of gravity and H_0, L_0, U_0, and f_0 denote a characteristic fluid depth, horizontal length, horizontal fluid velocity, and Coriolis parameter, respectively. So, for example, $f(\boldsymbol{x}) = f_{\mathrm{ph}}(\boldsymbol{x})/f_0$, where f_{ph} denotes the Coriolis parameter in physical, unscaled variables, and f_0 denotes a suitable defined average.

In this paper, we study the semigeostrophic regime, where ε is small and $B = \mathrm{O}(\varepsilon)$, while fluctuations of the fluid depth may be comparable to the layer depth. We also consider the fluid to be stationary at infinity, so that $\boldsymbol{u}$ vanishes and $h = b = b_0$ are constant outside some unspecified compact subdomain. For simplicity, we set $B \equiv \varepsilon$ and $h_0 \equiv 1$. Then, the velocity to leading order in ε is given by the *geostrophic balance* relation

$$\boldsymbol{u}_{\mathrm{G}} = f^{-1}\,\boldsymbol{\nabla}^\perp \tilde{h}\,, \tag{3}$$

where $\tilde{h} = h - b$ denotes the surface elevation.

The rotating shallow water equations (1) can be derived from a variational principle as follows. We write $\boldsymbol{a}$ to denote particle labels and $\boldsymbol{\eta}$ to denote the flow of $\boldsymbol{u}$. Then, the fluid particle initially at location $\boldsymbol{a}$ is found at location $\boldsymbol{x} = \boldsymbol{\eta}(\boldsymbol{a}, t)$ at time t, and $\boldsymbol{\eta}(\,\cdot\,, t)$ is the diffeomorphism satisfying

$$\dot{\boldsymbol{\eta}} = \boldsymbol{u} \circ \boldsymbol{\eta} \qquad \text{with} \qquad \boldsymbol{\eta}(\,\cdot\,, 0) = \mathrm{Id}\,. \tag{4}$$

Here, the dot denotes the time derivative and $\boldsymbol{u} \circ \boldsymbol{\eta}$ is a shorthand for $\boldsymbol{u}(\boldsymbol{\eta}(\boldsymbol{a}, t), t)$. The layer depth h is related to the flow via

$$h = \frac{1}{J_{\boldsymbol{\eta}}} \circ \boldsymbol{\eta}^{-1}\,, \tag{5}$$

where $J_{\boldsymbol{\eta}} \equiv \det \boldsymbol{\nabla} \boldsymbol{\eta}$ is the Jacobian of $\boldsymbol{\eta}$. Then, h satisfies the continuity equation (1b) by Liouville's theorem, and the momentum equation (1a) is equivalent to stationarity of the action

$$S_{\mathrm{RSW}} = \int_{t_1}^{t_2} L_{\mathrm{RSW}}(\boldsymbol{\eta}, \dot{\boldsymbol{\eta}})\,\mathrm{d}t \tag{6}$$

with respect to variations of the flow map $\boldsymbol{\eta}$ which vanish at the temporal end points, and where the rotating shallow water Lagrangian is given by

$$\begin{aligned} L_{\mathrm{RSW}} &= \int_{\mathbb{R}^2} \boldsymbol{R} \circ \boldsymbol{\eta} \cdot \dot{\boldsymbol{\eta}} + \frac{\varepsilon}{2}\, |\dot{\boldsymbol{\eta}}|^2 - \frac{1}{2h}\, \tilde{h}^2 \circ \boldsymbol{\eta}\,\mathrm{d}\boldsymbol{a} \\ &= \int_{\mathbb{R}^2} h \left(\boldsymbol{R} \cdot \boldsymbol{u} + \frac{\varepsilon}{2}\, |\boldsymbol{u}|^2 \right) - \frac{1}{2}\, \tilde{h}^2\,\mathrm{d}\boldsymbol{x}\,. \end{aligned} \tag{7}$$

Here, the vector field $\boldsymbol{R}$ is a vector potential for the Coriolis parameter, i.e., $\boldsymbol{\nabla}^\perp \cdot \boldsymbol{R} = f$. Variational principles for fluids go back to Herivel (1955); in this form, including the variable bottom topography term, it appears already in Allen and Holm (1996).

Using Euler–Poincaré reduction, the variational principle can be stated purely in terms of the Eulerian quantities $\boldsymbol{u}$ and h whose variations $\delta \boldsymbol{u}$ and δh are related to variations of the flow map

$$\delta \boldsymbol{\eta} \equiv \boldsymbol{w} \circ \boldsymbol{\eta} \tag{8}$$

by the Lin constraints (Bretherton 1970)

$$\delta \boldsymbol{u} = \dot{\boldsymbol{w}} + \boldsymbol{\nabla} \boldsymbol{w}\, \boldsymbol{u} - \boldsymbol{\nabla} \boldsymbol{u}\, \boldsymbol{w} = \dot{\boldsymbol{w}} + [\boldsymbol{u}, \boldsymbol{w}]\,, \tag{9a}$$

$$\delta h + \boldsymbol{\nabla} \cdot (h \boldsymbol{w}) = 0\,. \tag{9b}$$

(We read vector fields as column vectors and $\boldsymbol{\nabla} \boldsymbol{w}$ as the matrix $(\partial_j w_i)_{ij}$, so that by the usual rules of matrix multiplication, $(\boldsymbol{\nabla} \boldsymbol{w}\, \boldsymbol{u})_i = \sum_j \partial_j w_i\, u_j$.) These relations express how variations in $\boldsymbol{u}$ and h relate to variations in the flow map as they appear in a general statement of Hamilton's principle for continuum mechanics. The first Lin constraint is obtained by equating $\delta \dot{\boldsymbol{\eta}}$ calculated from (4) and (8), respectively; the second is obtained by taking the variation in (5) and applying Liouville's theorem.

The invariance of the Lagrangian under time translations implies conservation of the energy

$$H_{\mathrm{RSW}} = \frac{1}{2} \int_{\mathbb{R}^2} \varepsilon\, h\, |\boldsymbol{u}|^2 + \tilde{h}^2\,\mathrm{d}\boldsymbol{x}\,, \tag{10}$$

while the invariance of the Lagrangian under particle relabeling implies the conservation of *potential vorticity*

$$q = \frac{f + \varepsilon\, \boldsymbol{\nabla}^\perp \cdot \boldsymbol{u}}{h} \tag{11}$$

along particle trajectories.

3. Variational principle for balance models

In the semigeostrophic regime, balance models filter inertia-gravity waves by replacing the momentum equation (1a) with a kinematic dependence of the fluid velocity on the mass configuration. Such a kinematic dependence can only arise via a Hamilton variational principle if the Lagrangian is affine in the velocity. (We use the term “affine” rather than “linear” to emphasize that a nonzero velocity-independent additive contribution must be part of the Lagrangian to obtain nontrivial Euler–Lagrange equations.) We further demand of any reasonable balance model that its Lagrangian is invariant under particle relabeling so that the model possesses, by the Noether theorem, an advected potential vorticity. This requirement is automatically satisfied by any function of the Eulerian velocity $\boldsymbol{u}$ and layer depth h. These two considerations lead us to seek balance model Lagrangians of the form

$$L_{\mathrm{BM}}(\boldsymbol{u}, h) = \int_{\mathbb{R}^2} h \left(\boldsymbol{F}(h) \cdot \boldsymbol{u} - G(h)\right) \mathrm{d}\boldsymbol{x} \,, \tag{12}$$

where $\boldsymbol{F} \circ \eta$ and $G \circ \eta$ are the canonical Lagrangian momentum and energy density, respectively. In the following, they will be given by formal power series in ε. Any Lagrangian of the form (12) is invariant under the particle relabeling transformation $\boldsymbol{\eta} \mapsto \boldsymbol{\eta} \circ \boldsymbol{\xi}$, where $\boldsymbol{\xi}$ is an arbitrary volume preserving diffeomorphism, as h in (5) remains unchanged under this transformation.

Before deriving concrete first order balance models in section 4, we state the Euler–Lagrange equations associated with L_{BM} for general $\boldsymbol{F}$ and G. These are defined as the stationary points of the action

$$S_{\mathrm{BM}}(\boldsymbol{u}, h) = \int_{t_1}^{t_2} L_{\mathrm{BM}}(\boldsymbol{u}, h) \, \mathrm{d}t \tag{13}$$

with respect to variations $\delta \boldsymbol{u}$ and δh subject to the Lin constraints (9) which vanish at the temporal end points. A somewhat lengthy routine calculation using (9), the details of which can be found in Oliver (2006), yields

$$\begin{aligned} \delta S_{BM} = \int_{t_1}^{t_2} \int_{\mathbb{R}^2} h\boldsymbol{w} \cdot \big[&\boldsymbol{\nabla}(\mathrm{D}\boldsymbol{F}^*(h) \cdot (h\boldsymbol{u})) + \mathrm{D}\boldsymbol{F}(h) \boldsymbol{\nabla} \cdot (h\boldsymbol{u}) - \boldsymbol{u}^\perp \, \boldsymbol{\nabla}^\perp \cdot \boldsymbol{F} \\ &- \boldsymbol{\nabla}(\mathrm{D}G^*(h) h) + \boldsymbol{\nabla} G \big] \, \mathrm{d}\boldsymbol{x} \, \mathrm{d}t = 0 \,, \end{aligned} \tag{14}$$

where $\boldsymbol{w}$ is an arbitrary vector field associated with the variation of the flow map via (8), $\mathrm{D}\boldsymbol{F}$ and $\mathrm{D}G$ denote the functional derivatives of $\boldsymbol{F}$ and G, respectively, and $\mathrm{D}\boldsymbol{F}^*$ and $\mathrm{D}G^*$ denote the corresponding formal L^2 adjoint operators, defined by

$$\langle \mathrm{D}\boldsymbol{F}(h)\phi, \boldsymbol{w} \rangle = \langle \phi, \mathrm{D}\boldsymbol{F}^*(h) \cdot \boldsymbol{w} \rangle \quad \text{and} \quad \langle \mathrm{D}G(h)\phi, \psi \rangle = \langle \phi, \mathrm{D}G^*(h)\psi \rangle \tag{15}$$

for arbitrary ϕ, ψ, and $\boldsymbol{w}$ in the respective operator domains, where $\langle \cdot \, , \cdot \rangle$ denotes the inner product for L^2 functions and vector fields, respectively.

Since $\boldsymbol{w}$ in (14) is arbitrary, the term in brackets must vanish pointwise, so that the Euler–Lagrange equations read

$$\boldsymbol{\nabla}(\mathrm{D}\boldsymbol{F}^*(h) \cdot (h\boldsymbol{u})) + \mathrm{D}\boldsymbol{F}(h) \boldsymbol{\nabla} \cdot (h\boldsymbol{u}) - \boldsymbol{u}^\perp \, \boldsymbol{\nabla}^\perp \cdot \boldsymbol{F} = \boldsymbol{\nabla}(\mathrm{D}G^*(h) h) + \boldsymbol{\nabla} G \,. \tag{16}$$

Formally, the continuity equation (1b), being equivalent to the definition of h in (5), and (16) form a closed system for the resulting dynamics.

Further, it is straightforward to verify that the potential vorticity

$$q = \frac{\boldsymbol{\nabla}^\perp \cdot \boldsymbol{F}(h)}{h} \tag{17}$$

is advected by the velocity field $\boldsymbol{u}$, i.e.,

$$\partial_t q + \boldsymbol{u} \cdot \boldsymbol{\nabla} q = 0 \,, \tag{18}$$

and that the energy

$$H_{\mathrm{BM}} = \left\langle \frac{\delta L_{\mathrm{BM}}}{\delta \boldsymbol{u}} , \boldsymbol{u} \right\rangle - L_{\mathrm{BM}} = \int_{\mathbb{R}^2} h\, G(h)\, \mathrm{d}\boldsymbol{x} \,, \tag{19}$$

where $\langle \cdot \,, \cdot \rangle$ is the pairing between the space of Eulerian velocities and its dual, is a constant of the motion. These two conservation laws can be obtained from Noether's theorem applied to the particle relabeling symmetry and the invariance under time translation, respectively. A more direct proof of (18) is obtained by taking the curl of (16), dividing through by h, and eliminating extra terms via the continuity equation (1b).

4. First order balance models

Our approach to derive balance models is based on finding a near-identity transformation that renders the formal asymptotic expansion of the shallow water Lagrangian with respect to Rossby number affine in the velocity. Affine Lagrangians are degenerate and imply a so-called Dirac constraint on the dynamics. We remark that the method allows for models of arbitrary accuracy, but the calculations become rather cumbersome already at $\mathrm{O}(\varepsilon^2)$. For this reason, we restrict our attention to $\mathrm{O}(\varepsilon^1)$ models.

To distinguish physical and transformed coordinates, we use the following convention. Quantities in "old" physical coordinates shall carry a subscript ε, while unadorned quantities denote their counterparts in the yet-to-be-determined computational coordinate system. Our ansatz is that the physical and computational flow maps are related by the transformation

$$\boldsymbol{\eta}_\varepsilon = \boldsymbol{\xi}_\varepsilon \circ \boldsymbol{\eta} \,, \tag{20}$$

where the diffeomorphism $\boldsymbol{\xi}_\varepsilon$ is generated by a vector field $\boldsymbol{v}$, i.e.,

$$\boldsymbol{\xi}_\varepsilon(\boldsymbol{x}, t) = \boldsymbol{x} + \varepsilon\, \boldsymbol{v}(\boldsymbol{x}, t) + \mathrm{O}(\varepsilon^2) \,. \tag{21}$$

First, we note that

$$(\boldsymbol{w}_1 + \varepsilon\, \boldsymbol{w}_2 + \mathrm{O}(\varepsilon^2)) \circ \boldsymbol{\xi}_\varepsilon^{-1} = \boldsymbol{w}_1 + \varepsilon\, \boldsymbol{w}_2 - \varepsilon\, \boldsymbol{\nabla} \boldsymbol{w}_1\, \boldsymbol{v} + \mathrm{O}(\varepsilon^2) \tag{22}$$

for arbitrary ε-independent vector fields $\boldsymbol{w}_1$ and $\boldsymbol{w}_2$, which can be proved by differentiating the left hand side with respect to ε. Then, we compute, using (4), (20), (21), and (22),

$$\begin{aligned} \boldsymbol{u}_\varepsilon = \dot{\boldsymbol{\eta}}_\varepsilon \circ \boldsymbol{\eta}_\varepsilon^{-1} &= \big(\dot{\boldsymbol{\xi}}_\varepsilon + \mathrm{D}\boldsymbol{\xi}_\varepsilon\, \boldsymbol{u}\big) \circ \boldsymbol{\eta} \circ \boldsymbol{\eta}_\varepsilon^{-1} \\ &= \big(\boldsymbol{u} + \varepsilon\, \dot{\boldsymbol{v}} + \varepsilon\, \boldsymbol{\nabla} \boldsymbol{v}\, \boldsymbol{u} + \mathrm{O}(\varepsilon^2)\big) \circ \boldsymbol{\xi}_\varepsilon^{-1} \\ &= \boldsymbol{u} + \varepsilon\, (\dot{\boldsymbol{v}} + [\boldsymbol{u}, \boldsymbol{v}]) + \mathrm{O}(\varepsilon^2) \,. \end{aligned} \tag{23}$$

The first order term in (23) can be seen as analogous to the Lin constraint (9a) and can be similarly derived by equating the mixed derivatives $\partial_\varepsilon(\partial_t \boldsymbol{\eta}_\varepsilon)$ and $\partial_t(\partial_\varepsilon \boldsymbol{\eta}_\varepsilon)$.

To obtain an expansion for h_ε, note that, by the chain rule,

$$J_{\boldsymbol{\eta}_\varepsilon} = J_{\boldsymbol{\eta}}\, (J_{\boldsymbol{\xi}_\varepsilon} \circ \boldsymbol{\eta}) \,. \tag{24}$$

Hence, by (5),

$$h_\varepsilon = \frac{1}{J_{\boldsymbol{\eta}_\varepsilon}} \circ \boldsymbol{\eta}_\varepsilon^{-1} = \frac{1}{J_{\boldsymbol{\eta}}\, (J_{\boldsymbol{\xi}_\varepsilon} \circ \boldsymbol{\eta})} \circ \boldsymbol{\eta}^{-1} \circ \boldsymbol{\xi}_\varepsilon^{-1} = \frac{h}{J_{\boldsymbol{\xi}_\varepsilon}} \circ \boldsymbol{\xi}_\varepsilon^{-1} \,. \tag{25}$$

Then, taking the Jacobian of (21), substituting the expansion

$$\frac{h}{J_{\boldsymbol{\xi}_\varepsilon}} = \frac{h}{1 + \varepsilon\, \boldsymbol{\nabla} \cdot \boldsymbol{v} + \mathrm{O}(\varepsilon^2)} = h - \varepsilon\, h\, \boldsymbol{\nabla} \cdot \boldsymbol{v} + \mathrm{O}(\varepsilon^2) \tag{26}$$

into (25), and using (22), we obtain

$$h_\varepsilon = h - \varepsilon\,\boldsymbol{\nabla}\cdot(h\boldsymbol{v}) + \mathrm{O}(\varepsilon^2)\,. \tag{27}$$

We remark that (27) is nothing but the Lin constraint (9b) with variations parametrized by ε.

We now insert (23) and (27) into the shallow water Lagrangian (7) to obtain

$$\begin{aligned}
L_{\mathrm{RSW}} &= \int_{\mathbb{R}^2} h_\varepsilon\,\bigl(\boldsymbol{R}\cdot\boldsymbol{u}_\varepsilon + \tfrac12\,\varepsilon\,|\boldsymbol{u}_\varepsilon|^2\bigr) - \tfrac12\,\tilde h_\varepsilon^2\,\mathrm{d}\boldsymbol{x} \\
&= \int_{\mathbb{R}^2} h\,\bigl[\boldsymbol{R}\cdot\boldsymbol{u} + \varepsilon\,\bigl(\boldsymbol{R}\cdot\dot{\boldsymbol{v}} + \boldsymbol{R}\cdot[\boldsymbol{u},\boldsymbol{v}] + \tfrac12\,|\boldsymbol{u}|^2\bigr)\bigr]\,\mathrm{d}\boldsymbol{x} \\
&\quad + \int_{\mathbb{R}^2} \varepsilon\,\boldsymbol{\nabla}\cdot(h\boldsymbol{v})\,(\tilde h - \boldsymbol{R}\cdot\boldsymbol{u}) - \tfrac12\,\tilde h^2\,\mathrm{d}\boldsymbol{x} + \mathrm{O}(\varepsilon^2) \\
&= \int_{\mathbb{R}^2} h\,\boldsymbol{R}\cdot\boldsymbol{u} - \tfrac12\,\tilde h^2\,\mathrm{d}\boldsymbol{x} + \varepsilon\int_{\mathbb{R}^2} h\,\bigl(f\,\boldsymbol{u}\cdot\boldsymbol{v}^\perp + \tfrac12\,|\boldsymbol{u}|^2 - \boldsymbol{v}\cdot\boldsymbol{\nabla}\tilde h\bigr)\,\mathrm{d}\boldsymbol{x} \\
&\quad + \varepsilon\,\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^2} h\,\boldsymbol{R}\cdot\boldsymbol{v}\,\mathrm{d}\boldsymbol{x} + \mathrm{O}(\varepsilon^2)\,,
\end{aligned} \tag{28}$$

where, as before, h denotes the layer depth and $\tilde h = h - b$ denotes the surface elevation. The last equality in (28) is due the continuity equation (1b) and the vector identity

$$\boldsymbol{R}\cdot[\boldsymbol{u},\boldsymbol{v}] + \boldsymbol{v}\cdot\boldsymbol{\nabla}(\boldsymbol{R}\cdot\boldsymbol{u}) - \boldsymbol{u}\cdot\boldsymbol{\nabla}(\boldsymbol{R}\cdot\boldsymbol{v}) = (\boldsymbol{\nabla}^\perp\cdot\boldsymbol{R})\,\boldsymbol{u}\cdot\boldsymbol{v}^\perp = f\,\boldsymbol{u}\cdot\boldsymbol{v}^\perp\,. \tag{29}$$

The perfect time derivative in (28) does not contribute to the variation of the action and can thus be discarded. We observe that any choice of the form

$$\boldsymbol{v} = \frac{1}{2f}\,\boldsymbol{u}^\perp + \boldsymbol{u}\,F_1(\boldsymbol{u},h,\boldsymbol{x}) + \boldsymbol{F}_2(h,\boldsymbol{x}) \tag{30}$$

will render L_{RSW} affine to first order. Here, we restrict our attention to the special case

$$\boldsymbol{v} = \frac{1}{2f}\,\boldsymbol{u}^\perp + \nu(\boldsymbol{x})\,\boldsymbol{u} + \tau(\boldsymbol{x})\,\boldsymbol{\nabla}\tilde h + \mu(\boldsymbol{x})\,\boldsymbol{\nabla}^\perp\tilde h\,, \tag{31}$$

i.e., the new terms are proportional or perpendicular-proportional to the first at the order of geostrophic balance. As we are dealing with spatially varying f, we allow for $\boldsymbol{x}$-dependence of the constants of proportionality, though. This choice of $\boldsymbol{v}$ has the advantage that it does not introduce terms with a different homogeneity from those already present in the shallow water Lagrangian. We further note that when $\tau = \frac12\,f^{-2}$ and $\nu = -f\mu$, the transformation vector field $\boldsymbol{v}$ vanishes up to terms of $\mathrm{O}(\varepsilon)$, which give $\mathrm{O}(\varepsilon^2)$ contribution to (28) and, therefore, can be neglected. We may then interpret the resulting balance model as having been written directly in physical coordinates.

Substituting (31) into (28) with its perfect time derivative term discarded, we find that the first order approximation to L_{RSW} reads

$$L = \int_{\mathbb{R}^2} h\,\biggl[\Bigl(\boldsymbol{R} + \varepsilon\,(\alpha\boldsymbol{\nabla}^\perp\tilde h - \beta\boldsymbol{\nabla}\tilde h)\Bigr)\cdot\boldsymbol{u} - \varepsilon\,\tau\,|\boldsymbol{\nabla}\tilde h|^2 - \frac{\tilde h^2}{2h}\biggr]\,\mathrm{d}\boldsymbol{x} \tag{32}$$

with

$$\alpha = f\tau + \frac{1}{2f} \qquad \text{and} \qquad \beta = f\mu + \nu\,. \tag{33}$$

The equations of motion are a special case of the abstract Euler–Lagrange equation (16): in the notation of section 3,

$$\boldsymbol{F}(h) = \boldsymbol{R} + \varepsilon\,(\alpha \boldsymbol{\nabla}^\perp \tilde{h} - \beta \boldsymbol{\nabla} \tilde{h}) \quad \text{and} \quad G(h) = \frac{\tilde{h}^2}{2h} + \varepsilon\,\tau\,|\boldsymbol{\nabla}\tilde{h}|^2\,, \tag{34}$$

so that, for an arbitrary scalar function ϕ,

$$\mathrm{D}\boldsymbol{F}(h)\phi = \varepsilon\,(\alpha \boldsymbol{\nabla}^\perp \phi - \beta \boldsymbol{\nabla}\phi)\,, \tag{35}$$

$$\mathrm{D}G(h)\phi = \left(\frac{\tilde{h}}{h} - \frac{1}{2}\frac{\tilde{h}^2}{h^2}\right)\phi + 2\,\varepsilon\,\tau\,\boldsymbol{\nabla}\tilde{h}\cdot\boldsymbol{\nabla}\phi\,, \tag{36}$$

and, for any vector field $\boldsymbol{w}$ and scalar field ψ,

$$\mathrm{D}\boldsymbol{F}^*(h)\cdot\boldsymbol{w} = \varepsilon\,\boldsymbol{\nabla}\cdot(\alpha\boldsymbol{w}^\perp + \beta\boldsymbol{w})\,, \tag{37}$$

$$\mathrm{D}G^*(h)\psi = \left(\frac{\tilde{h}}{h} - \frac{1}{2}\frac{\tilde{h}^2}{h^2}\right)\psi - 2\,\varepsilon\,\boldsymbol{\nabla}\cdot(\tau\psi\boldsymbol{\nabla}\tilde{h})\,. \tag{38}$$

Substituting (34–38) into (16), using the vector identity

$$\boldsymbol{\nabla}\boldsymbol{\nabla}\cdot\boldsymbol{w} + \boldsymbol{\nabla}^\perp\boldsymbol{\nabla}^\perp\cdot\boldsymbol{w} = \Delta\boldsymbol{w}\,, \tag{39}$$

and rearranging terms, we obtain

$$\Lambda_h \boldsymbol{u} = \boldsymbol{\nabla}^\perp\big[\tilde{h} - \varepsilon\,\big(2\,\boldsymbol{\nabla}\cdot(\tau h\boldsymbol{\nabla}\tilde{h}) - \tau\,|\boldsymbol{\nabla}\tilde{h}|^2\big)\big]\,, \tag{40}$$

where

$$\begin{aligned}\Lambda_h \boldsymbol{u} = {} & \big[\tilde{f} + \varepsilon\,\big(\boldsymbol{\nabla}\cdot(\alpha\boldsymbol{\nabla}h) - \boldsymbol{\nabla}^\perp\beta\cdot\boldsymbol{\nabla}h\big)\big]\,\boldsymbol{u} + \varepsilon\,(\boldsymbol{\nabla}\alpha + \boldsymbol{\nabla}^\perp\beta)\,\boldsymbol{\nabla}\cdot(h\boldsymbol{u}) \\ & + \varepsilon\,\big(\boldsymbol{\nabla}(h\boldsymbol{u}\cdot\boldsymbol{\nabla}\alpha) - \boldsymbol{\nabla}^\perp(h\boldsymbol{u}\cdot\boldsymbol{\nabla}\beta) - \Delta(\alpha h\boldsymbol{u})\big)\,,\end{aligned} \tag{41a}$$

$\tilde{f}$ denotes the effective Coriolis parameter

$$\tilde{f} = f - \varepsilon\,\boldsymbol{\nabla}\cdot(\alpha\boldsymbol{\nabla}b) + \varepsilon\,\boldsymbol{\nabla}^\perp\beta\cdot\boldsymbol{\nabla}b\,, \tag{41b}$$

and $\tilde{h}$ denotes the surface elevation

$$\tilde{h} = h - b\,. \tag{41c}$$

A complete set of equations of motion consists of the momentum equation (40) complemented with the continuity equation (1b). Note that one recovers the expression for the geostrophic velocity (3) as the zero order balance in (40).

Inserting (34) into (17), we obtain the expression for the potential vorticity,

$$q = \frac{f + \varepsilon\,\boldsymbol{\nabla}\cdot(\alpha\boldsymbol{\nabla}\tilde{h}) - \varepsilon\,\boldsymbol{\nabla}^\perp\beta\cdot\boldsymbol{\nabla}\tilde{h}}{h}\,. \tag{42}$$

Finally, inserting (34) into (19), we obtain the Hamiltonian

$$H = \int_{\mathbb{R}^2} \tfrac{1}{2}\,\tilde{h}^2 + \varepsilon\,\tau\,h\,|\boldsymbol{\nabla}\tilde{h}|^2\,\mathrm{d}\boldsymbol{x}\,. \tag{43}$$

5. Potential vorticity formulation

Special significance among balance models have those that can be formulated in terms of potential vorticity advection and inversion. Full PV inversion requires solving the pair of second order differential equations (42) and (40), where the solution of the first equation becomes a parameter of the second. We claim that when α, $\tilde{f}$, and the initial PV data are uniformly positive, then the two equations are uniformly elliptic. To see this, let us first rewrite (42) as

$$L_q(h-1) = \tilde{f} - q\,, \tag{44}$$

where L_q is a linear operator such that for any scalar function ϕ

$$L_q\phi = q\,\phi - \varepsilon\,\boldsymbol{\nabla}\cdot(\alpha\boldsymbol{\nabla}\phi) + \varepsilon\,\boldsymbol{\nabla}^{\perp}\beta\cdot\boldsymbol{\nabla}\phi\,. \tag{45}$$

Note that we assumed that $h=1$ outside of a compact subdomain. This can be achieved by ensuring that $\tilde{f} = q$ initially outside of a compact subdomain. We may then analyze (44) as a homogeneous Dirichlet problem on a sufficiently large bounded domain.

When α is uniformly positive, i.e.,

$$\alpha(\boldsymbol{x}) \geq \alpha_0 > 0 \tag{46}$$

for some constant α_0, L_q is uniformly elliptic. (We note that when $\alpha = 0$, we still obtain a particularly simply PV formulation, but the resulting system is ill-posed; cf. comments in section 6.) Further, when q is uniformly positive, then the zero-order coefficient of L_q is positive and, under the assumption of sufficient regularity of all quantities involved, classical Schauder theory for second order elliptic equations with nonconstant coefficients implies that (44) has a unique solution (see, e.g., Gilbarg and Trudinger 1983, Chapter 6.3). To proceed, we recall the classical Hopf (1927) maximal principle (Pucci and Serrin 2004, Theorems 2.1 and 2.2).

(Hopf Maximum Principle.) *Let u be a C^2 function satisfying the differential inequality*

$$\sum_{i,j} a_{ij}\,\frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_i b_i\,\frac{\partial u}{\partial x_i} + c\,u \geq 0 \quad (\leq 0) \tag{47}$$

in a domain $\Omega \subset \mathbb{R}^n$, where the symmetric matrix $a_{ij} = a_{ij}(\boldsymbol{x})$ is locally uniformly negative definite in Ω, the coefficients a_{ij}, $b_i = b_i(\boldsymbol{x})$, and $c = c(\boldsymbol{x})$ are locally bounded, and c is a non-negative function on Ω. If u takes a non-positive minimum (non-negative maximum) value M in Ω, then $u \equiv M$.

As an immediate consequence, we obtain the pointwise bounds

$$0 < \inf_{\boldsymbol{x}\in\mathbb{R}^2} \frac{\tilde{f}}{q} \leq h \leq \sup_{\boldsymbol{x}\in\mathbb{R}^2} \frac{\tilde{f}}{q}\,. \tag{48}$$

Indeed, rewriting (44) as

$$L_q\biggl(h - \inf_{\boldsymbol{x}\in\mathbb{R}^2} \frac{\tilde{f}}{q}\biggr) = q\,\biggl(\frac{\tilde{f}}{q} - \inf_{\boldsymbol{x}\in\mathbb{R}^2} \frac{\tilde{f}}{q}\biggr) \geq 0\,, \tag{49}$$

noting that $h - \tilde{f}/q \to 0$ as $|\boldsymbol{x}| \to \infty$, and using the Hopf principle, we obtain the lower bound in (48). The upper bound follows from the corresponding argument for $h - \sup_{\boldsymbol{x}} \tilde{f}/q$.

Since potential vorticity is advected, $q_- \equiv \inf_{\boldsymbol{x}} q(\boldsymbol{x})$ and $q_+ \equiv \sup_{\boldsymbol{x}} q(\boldsymbol{x})$ are constants of the motion. Hence, the time dependent estimate (48) can be replaced by a weaker global bound

$$0 < \frac{\tilde{f}_-}{q_+} \leq h \leq \frac{\tilde{f}_+}{q_-}\,. \tag{50}$$

Let us now turn our attention to the momentum equation (40), the second stage of the PV inversion. First, we introduce the necessary functional spaces. As customary, $L^p(\mathbb{R}^2)$ for $1 \leq p \leq \infty$ denotes the Lebesgue space of functions integrable to power p over $\mathbb{R}^2$. For $k \geq 0$, the Sobolev space $W^{k,p}(\mathbb{R}^2)$ is the space of functions whose partial distributional derivatives up to order k belong to $L^p(\mathbb{R}^2)$, endowed with the norm

$$\|w\|_{W^{k,p}} = \left(\sum_{|\gamma|\leq k} \int_{\mathbb{R}^2} |\mathrm{D}^\gamma w|^p \, \mathrm{d}\boldsymbol{x} \right)^{1/p} . \tag{51}$$

We write $H^k(\mathbb{R}^2)$ and $H^{-k}(\mathbb{R}^2)$ to denote the Hilbert space $W^{k,2}(\mathbb{R}^2)$ and its functional-analytic dual, respectively. Finally, we write $\boldsymbol{w} \in W^{k,p}(\mathbb{R}^2)$ for vector valued functions provided each component of $\boldsymbol{w}$ belongs to $W^{k,p}(\mathbb{R}^2)$.

As we have shown, αh is uniformly positive, hence Λ_h is uniformly elliptic and its unique invertibility is equivalent to injectivity by the Fredholm alternative (Gilbarg and Trudinger 1983). The simplest and most natural sufficient condition for injectivity is positivity, i.e.,

$$\langle \Lambda_h \boldsymbol{w}, \boldsymbol{w} \rangle > 0 \tag{52}$$

for any vector field $\boldsymbol{w} \in H^1(\mathbb{R}^2)$, where $\langle \cdot, \cdot \rangle$ denotes the pairing between $H^1(\mathbb{R}^2)$ and $H^{-1}(\mathbb{R}^2)$.

Let us assume, for simplicity, that β is constant across the domain. Then all the terms involving β in the momentum equation and the definition of potential vorticity vanish. Integrating by parts, we obtain

$$\begin{aligned} \int_{\mathbb{R}^2} \boldsymbol{w} \cdot \Lambda_h \boldsymbol{w} \, \mathrm{d}\boldsymbol{x} &= \int_{\mathbb{R}^2} \left[\tilde{f} + \varepsilon \, \boldsymbol{\nabla} \cdot (\alpha \boldsymbol{\nabla} h) - \tfrac{1}{2} \, \varepsilon \, \Delta(\alpha h) \right] |\boldsymbol{w}|^2 \, \mathrm{d}\boldsymbol{x} \\ &\quad + \varepsilon \int_{\mathbb{R}^2} \alpha h \, |\boldsymbol{\nabla} \boldsymbol{w}|^2 + (\boldsymbol{\nabla} \alpha \cdot \boldsymbol{w}) \, (\boldsymbol{\nabla} h \cdot \boldsymbol{w}) \, \mathrm{d}\boldsymbol{x} \, . \end{aligned} \tag{53}$$

Formula (42) implies that

$$\tilde{f} + \frac{\varepsilon}{2} \, \boldsymbol{\nabla} \cdot (\alpha \boldsymbol{\nabla} h) = \frac{qh + \tilde{f}}{2} \, . \tag{54}$$

Substituting (54) into (53), we can write the invertibility condition in the form

$$\int_{\mathbb{R}^2} \tfrac{1}{2} \left[\tilde{f} + qh - \varepsilon \, \boldsymbol{\nabla} \cdot (h \boldsymbol{\nabla} \alpha) \right] |\boldsymbol{w}|^2 + \varepsilon \left[\alpha h \, |\boldsymbol{\nabla} \boldsymbol{w}|^2 + (\boldsymbol{\nabla} \alpha \cdot \boldsymbol{w}) \, (\boldsymbol{\nabla} h \cdot \boldsymbol{w}) \right] \mathrm{d}\boldsymbol{x} > 0 \, . \tag{55}$$

If α is a constant, the solvability condition (55) holds trivially. In the general case, it can be replaced by the stronger requirement

$$3 \, \varepsilon \, |\boldsymbol{\nabla} h| \, |\boldsymbol{\nabla} \alpha| \leq qh + \tilde{f} - \varepsilon \, h \, \Delta \alpha \, , \tag{56}$$

which arises via the Cauchy–Schwarz inequality applied to all terms in (55) that involve first order derivatives on α and h.

Once solvability is guaranteed, either by (52), (55), or by the stronger condition (56), we may ask for the regularity of the solution. Suppose that the coefficients α, β, τ, b, and $\tilde{f}$ are smooth. Then, by standard elliptic regularity theory for second order operators (see, e.g., Gilbarg and Trudinger 1983), a potential vorticity defect $\tilde{q} \equiv q - \tilde{f} \in H^m(\mathbb{R}^2)$ in equation (44) implies that $(h-1) \in H^{m+2}(\mathbb{R}^2)$. This implies that the right hand expression of (40), the second stage of the inversion, is of class $H^{m-1}(\mathbb{R}^2)$ in the general case, and of class $H^{m+1}(\mathbb{R}^2)$ when $\tau = 0$. (For $m > 0$, this is obvious since $H^{m+1}(\mathbb{R}^2)$ is a topological algebra, i.e., the norm of products is bounded above by the product of the norms. For $m = 0$, the proof is more involved, requiring the use of the Gagliardo–Nirenberg inequality and uniform bound on h given by (50).) Then, by elliptic regularity once again, $\boldsymbol{u} \in H^{m+1}(\mathbb{R}^2)$ in the general case and $\boldsymbol{u} \in H^{m+3}(\mathbb{R}^2)$ when $\tau = 0$. Following Çalik, Oliver and Vasylkevych (2012), it is possible to show that the same pattern of regularity holds more generally on the scale of spaces $W^{m,p}(\mathbb{R}^2)$ for $p \in (0, \infty)$ fixed.

The equivalence of the vorticity formulation of the balance model, comprised of equations (18), (44), and (40), and the $\boldsymbol{u}$-h formulation of the model, equations (40) and (1b), can be made rigorous by noting that differentiation of (42) and the use of (40) yields

$$\partial_t q + \boldsymbol{u} \cdot \boldsymbol{\nabla} q = -h^{-1} \, L_q[\partial_t h + \boldsymbol{\nabla} \cdot (h\boldsymbol{u})] \,. \tag{57}$$

When the initial PV data is uniformly positive, it remains so for all times as it evolves by scalar advection. Then uniform positivity of α implies that L_q is invertible at all times, and that h is uniformly positive and bounded. Hence, (57) implies the equivalence of continuity equation (1b) and PV advection.

We finally note that the uniform positivity requirements for the PV data and the effective Coriolis parameter $\tilde{f}$ are physically natural and consistent with the chosen scaling.

6. Special cases

We now discuss several distinct choices for the parameters ν, τ, and μ in our transformation (31). Let us recall that, when the Coriolis parameter is constant, the family of generalized LSG models in Oliver (2006) interpolates between Salmon's L_1 and LSG models, with one special model in the middle for which the PV inversion is of third order. Moreover, for all models in this family, the solvability condition (52) is satisfied provided that bottom topography variations are not too large. Salmon's L_1 model is distinguished by the fact that the transformation from physical to computational coordinates is the identity transformation up to terms at $\mathrm{O}(\varepsilon^2)$, which are beyond the formal order of accuracy of the model itself.

For a non-constant Coriolis parameter, such an alignment of features no longer takes place: one has to choose between the family interpolating between L_1 and LSG models (the model where PV inversion gains three derivatives belongs in here) and an alternative family for which the equations of motion and the solvability condition take a particularly simple form. This will be detailed below.

In all of the cases considered, we set $\mu = \nu = 0$ so that $\beta = 0$. It appears that nonzero values for these parameters may only be of use when deriving equations on bounded domains.

6.1. *Salmon's L_1 model*

The L_1 model as originally proposed by Salmon (1985) is derived by appropriately substituting the geostrophic velocity (3) as a constraint into an extended form of the Hamilton principle. The resulting Lagrangian reads

$$L_1 = \int_{\mathbb{R}^2} h \left(\boldsymbol{R} + \varepsilon \, \boldsymbol{u}_{\mathrm{G}} \right) \cdot \boldsymbol{u} - \frac{1}{2} \, \varepsilon \, h \, |\boldsymbol{u}_{\mathrm{G}}|^2 - \frac{1}{2} \, \tilde{h}^2 \, \mathrm{d}\boldsymbol{x} \,. \tag{58}$$

In the setting of section 3, we obtain this very model by choosing

$$\tau = \frac{1}{2f^2} \qquad \text{so that} \qquad \alpha = \frac{1}{f} \,. \tag{59}$$

In this case, substituting the leading order balance relation (40) into the expression for $\boldsymbol{v}$, equation (31), shows that the transformation vector field $\boldsymbol{v}$ vanishes to $\mathrm{O}(\varepsilon)$. Hence, the transformation from physical to computational coordinates is the identity transformation up to terms of $\mathrm{O}(\varepsilon^2)$.

The momentum equation retains its general form with $\alpha = f^{-1}$, $\beta = 0$,

$$\begin{aligned} \Lambda_h \boldsymbol{u} = \left[\tilde{f} + \varepsilon \, \left(\boldsymbol{\nabla} \cdot (\alpha \boldsymbol{\nabla} h) \right) \right] \boldsymbol{u} + \varepsilon \, \boldsymbol{\nabla} \alpha \boldsymbol{\nabla} \cdot (h\boldsymbol{u}) \\ + \varepsilon \left(\boldsymbol{\nabla}(h\boldsymbol{u} \cdot \boldsymbol{\nabla} \alpha) - \Delta(\alpha h \boldsymbol{u}) \right) , \end{aligned} \tag{60}$$

potential vorticity

$$q = \frac{\tilde{f} + 2\varepsilon \, \boldsymbol{\nabla} \cdot (f^{-1} \boldsymbol{\nabla} \tilde{h})}{h} \,, \tag{61}$$

and solvability condition (55) or (56).

When q and f are uniformly positive,

$$\sup_{\boldsymbol{x} \in \mathbb{R}^2} |\boldsymbol{\nabla} h| \leq \varepsilon^{-2/3} \, C \,, \tag{62}$$

where C depends only on the maximum and minimum value of q. Therefore, when the effective Coriolis parameter and the initial potential vorticity are uniformly positive and ε is small enough, the solvability condition is satisfied for all times. However, it appears difficult to get practical bounds on ε. Moreover, solvability at time $t = 0$ does not necessarily imply solvability for later times.

The proof of (62) is based on L^p-regularity of second order elliptic operators (e.g. Chen and Wu 1991, Theorem 5.4), which yields

$$\|h - 1\|_{W^{2,4}} \leq C_1 \left(\varepsilon^{-1} \, \|\tilde{f} - q\|_{L^4} + \|h - 1\|_{L^4} \right) \leq \varepsilon^{-1} \, C_2 \,. \tag{63}$$

Inserting this estimate into the Gagliardo–Nirenberg inequality

$$\|h - 1\|_{W^{1,\infty}} \leq C_3 \, \|h - 1\|_{W^{2,4}}^{2/3} \, \|h - 1\|_{L^\infty}^{1/3} \tag{64}$$

and recalling the maximum principle for h, we obtain (62). Similar arguments have been used by Çalik, Oliver and Vasylkevych (2012).

6.2. *Salmon's LSG model*

The large-scale semigeostrophic (LSG) equations are a second model proposed by Salmon (1985) as a Hamiltonian simplification of the L_1 model. In our setting, they are obtained by choosing $\tau = -1/(2f^2)$ so that $\alpha = 0$. The resulting momentum equation reads

$$\boldsymbol{u} = f^{-1} \, \boldsymbol{\nabla}^\perp \big[\tilde{h} - \varepsilon \left(2 \, \boldsymbol{\nabla} \cdot (\tau h \boldsymbol{\nabla} \tilde{h}) - \tau \, |\boldsymbol{\nabla} \tilde{h}|^2 \right) \big] \,. \tag{65}$$

Unfortunately, this model appears ill posed since (65) suggests that the advecting velocity field is less smooth than the advected PV. On the other hand, in general the advected field can be at most as smooth as the advecting vector field. So unless there is some undiscovered very special structure, ill-posedness results. This is confirmed by simple numerical tests in a space-periodic setting.

6.3. *Generalized LSG models*

When f is non-constant, there are at least two natural generalizations of the one-parameter family of models derived by Oliver (2006). First, we can set

$$\tau = \frac{\lambda}{f^2} \qquad \text{so that} \qquad \alpha = \frac{\lambda + \frac{1}{2}}{f} \,. \tag{66}$$

Then $\lambda = \frac{1}{2}$ and $\lambda = -\frac{1}{2}$ correspond to the L_1 and the LSG models discussed above; when $\lambda = 0$, we obtain a model for which the velocity field is three derivatives smoother than the potential vorticity. However, the entire family shares the difficulty arising from the solvability condition with the L_1 case.

Second, we can choose

$$\tau = \frac{\lambda + \frac{1}{2}}{f} - \frac{1}{2f^2} \qquad \text{so that} \qquad \alpha = \lambda + \tfrac{1}{2} \,. \tag{67}$$

For this family, the momentum equation (40) and the expression for the potential vorticity (42) take a particularly simple form, namely

$$\big(\tilde{f} - \varepsilon\,(\lambda + \tfrac{1}{2})\,(h\,\Delta + 2\,\boldsymbol{\nabla} h \cdot \boldsymbol{\nabla})\big)\boldsymbol{u} = \boldsymbol{\nabla}^{\perp}\big[\tilde{h} - \varepsilon\,\big(2\,\boldsymbol{\nabla}\cdot(\tau h \boldsymbol{\nabla}\tilde{h}) - \tau\,|\boldsymbol{\nabla}\tilde{h}|^2\big)\big] \tag{68}$$

and

$$q\,h = \tilde{f} + \varepsilon\,(\lambda + \tfrac{1}{2})\,\Delta h\,. \tag{69}$$

Since α is constant, the solvability condition reduces to the condition that the effective Coriolis parameter $\tilde{f} = f - \varepsilon\,(\lambda + \frac{1}{2})\,\Delta b$ is uniformly positive. Then, as we have argued in the previous section, equations (69) and (68) can be solved simultaneously for h and $\boldsymbol{u}$, respectively, at all times provided the initial PV data is uniformly positive and $\lambda + \frac{1}{2} > 0$. Under these assumptions, the full vorticity inversion as a nonlinear operator on q gains one derivative in Sobolev space. Hence, we expect that this second family of models is globally well posed without the need for additional solvability conditions, with a proof along the lines of Çalik, Oliver and Vasylkevych (2012).

We further note that the family (67) does not generally include the case $\tau = 0$ when extra regularity can be gained, nor the case $\tau = \frac{1}{2}\,f^{-2}$ when the transformation can be considered to be the identity to $\mathrm{O}(\varepsilon)$. In other words, we are trading the necessity to explicitly transform between coordinate systems for a robust solvability condition.

7. Generalized LSG as a constrained system

In the following, we give a brief account on the parallels and differences between Salmon's approach and our approach from the point of view of constrained Hamiltonian systems. Salmon (1985, 1996) viewed the derivation of Hamiltonian balance models as imposing a velocity constraint onto the variational principle. This point of view has subsequently been used by other authors (Holm 1996, Wunderer 2001, McIntyre and Roulstone 2002, Vanneste and Bokhove 2002). In our framework, the resulting degenerate Lagrangian can be reinterpreted as a constrained Hamiltonian system. In the following, we will make this link explicit.

First, we use the Legendre transform

$$\boldsymbol{p} = \frac{\delta L_{RSW}}{\delta \boldsymbol{u}} = h\,(\boldsymbol{R} + \varepsilon\,\boldsymbol{u})\,, \tag{70}$$

to write out the phase-space counterpart of the shallow water variational principle corresponding to (7), namely

$$\delta \int_{t_1}^{t_2} \langle \boldsymbol{p}, \boldsymbol{u} \rangle - H_{\mathrm{RSW}}(\boldsymbol{p}, h)\,\mathrm{d}t = 0\,, \tag{71}$$

with respect to independent variations $\delta\boldsymbol{p}$ and $\delta\boldsymbol{\eta}$, fixed at the temporal endpoints, where $\delta\boldsymbol{\eta}$ induces the variations $\delta\boldsymbol{u}$ and δh via the Lin constraints (9). The Hamiltonian $H_{\mathrm{RSW}}(\boldsymbol{p}, h)$ is obtained by solving (70) for $\boldsymbol{u}$ and inserting the result into the expression for the energy (10).

Imposing a momentum-configuration constraint of the form

$$\boldsymbol{p}^{\mathrm{c}} \equiv h\,(\boldsymbol{R} + \varepsilon\,\boldsymbol{u}^{\mathrm{c}}(h)) = \mathcal{F}(h) \tag{72}$$

into the variational principle (71) leads to an affine Lagrangian

$$L^{\mathrm{c}} = \langle \mathcal{F}(h), \boldsymbol{u} \rangle - H_{\mathrm{RSW}}(\mathcal{F}(h), h)\,. \tag{73}$$

Comparing (73) with (58), we observe that the L_1 model corresponds to the choice $\boldsymbol{u}^{\mathrm{c}} = \boldsymbol{u}_G$.

We must stress that all of the above takes place in physical coordinates, and should therefore be read with an ε subscript when comparing with our transformational approach. The generalized LSG equations can be similarly interpreted as imposing the momentum-configuration constraint

$$\boldsymbol{p}^{\mathrm{c}} = h\,(\boldsymbol{R} + \varepsilon\,\boldsymbol{u}^{\mathrm{GLSG}}) \tag{74}$$

with

$$\boldsymbol{u}^{\mathrm{GLSG}} = \alpha\boldsymbol{\nabla}^{\perp}\tilde{h} - \beta\boldsymbol{\nabla}\tilde{h} \tag{75}$$

onto the unconstrained Hamiltonian in transformed coordinates,

$$H_{\mathrm{uc}}(u,h) = \int_{\mathbb{R}^2} h\,\varepsilon\,\bigl(|\boldsymbol{u}|^2 - |\boldsymbol{u}^{\mathrm{GLSG}}|^2 + \tau\,|\boldsymbol{\nabla}\tilde{h}|^2\bigr) + \tfrac{1}{2}\,\tilde{h}^2\,\mathrm{d}\boldsymbol{x}\,. \tag{76}$$

To determine the constraint in physical coordinates implied by this construction, we must match

$$\int_{t_1}^{t_2} \langle \boldsymbol{p}_\varepsilon^{\mathrm{c}}, \boldsymbol{u}_\varepsilon\rangle\,\mathrm{d}t = \int_{t_1}^{t_2} \langle \boldsymbol{p}^{\mathrm{c}}, \boldsymbol{u}\rangle\,\mathrm{d}t\,, \tag{77}$$

fully expanded through to $\mathrm{O}(\varepsilon)$. A direct but lengthy calculation, based on identities similar to the ones used in (28), yields

$$\boldsymbol{u}_\varepsilon^{\mathrm{c}} = \boldsymbol{u}^{\mathrm{GLSG}} - f\,\boldsymbol{v}^{\perp} + \mathrm{O}(\varepsilon) = \boldsymbol{u}_G + \mathrm{O}(\varepsilon)\,, \tag{78}$$

as should be expected from any consistently constructed balance model, and $H_{\mathrm{RSW}} = H_{\mathrm{uc}} + \mathrm{O}(\varepsilon^2)$.

8. Concluding remarks

We have shown that the transformational approach of Oliver (2006) for deriving Hamiltonian balance models extends naturally to situations where the Coriolis parameter is spatially varying. The resulting balance models can, by construction, be described in terms of PV advection and inversion. However, the choices to be made are more subtle than in the case of a constant Coriolis parameter. In particular, while we can derive a model for which the PV inversion satisfies a robust solvability condition and the balance relation is no more complicated than for the L_1 model with constant f, this model is necessarily posed, unlike the L_1 model with constant f, in a coordinate system arising through $\mathrm{O}(\varepsilon)$ changes of variables. If, on the other hand, we seek a transformation which is negligible up to terms of $\mathrm{O}(\varepsilon^2)$—terms beyond the formal order of accuracy of the model—we lose simplicity of the balance relation and we lose robust solvability. Similarly, the model where PV inversion gains the maximum possible three derivatives has a non-robust solvability condition whenever the Coriolis parameter is spatially varying. Thus, we believe that the family of balance models characterized by (67), which is robustly solvable, whose balance relation is relatively simple, and which has not been suggested previously, is the most promising and warrants further study.

Our main restriction is that the derivation and the PV invertibility analysis are confined to an extra-equatorial region and can not easily be extended across the equator. For instance, if the Coriolis parameter has zeros, the transformational vector field (31) is necessarily singular. On the other hand, no further restrictions are placed on Coriolis parameter or on the bottom topography except those necessary to guarantee PV invertibility. In particular, order one variations in the scaled Coriolis parameter are allowed, so that the new models are potentially useful for simulations encompassing large latitude ranges.

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