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Global well-posedness for the generalized large-scale semigeostrophic equations

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Abstract

We prove existence and uniqueness of global classical solutions to the generalized large-scale semigeostrophic equations with periodic boundary conditions. This family of Hamiltonian balance models for rapidly rotating shallow water includes the L_1 model derived by R. Salmon in 1985 and its 2006 generalization by the second author. The results are, under the physical restriction that the initial potential vorticity is positive, as strong as those available for the Euler equations of ideal fluid flow in two dimensions. Moreover, we identify a special case in which the velocity field is two derivatives smoother in Sobolev space as compared to the general case.

Our results are based on careful estimates which show that, although the potential vorticity inversion is nonlinear, bounds on the potential vorticity inversion operator remain linear in derivatives of the potential vorticity. This permits the adaptation of an argument based on elliptic L^p theory, proposed by Yudovich in 1963 for proving existence and uniqueness of weak solutions for the two-dimensional Euler equations, to our particular nonlinear situation.

1. Introduction

We prove the existence of global classical solutions for a family of generalized large scale semigeostrophic (LSG) equations with periodic boundary conditions in two dimensions. The model can be formulated as an advection equation for a scalar potential vorticity $q = q(x,t)$ by a two dimensional velocity field $u = u(x,t)$,

$$\partial_t q + u \cdot \nabla q = 0\,, \tag{1a}$$

$$u = K(q)\,, \tag{1b}$$

where K is the nonlinear operator implicitly defined via

$$(q - \sigma \Delta) h = 1 \,, \tag{2a}$$

$$\bigl(1 - \sigma \, (h \, \Delta + 2 \, \nabla h \cdot \nabla)\bigr) u = \nabla^\perp \bigl(h - \mu \, (2 \, h \, \Delta h + |\nabla h|^2)\bigr) \,. \tag{2b}$$

Here, $\sigma > 0$ and μ are real parameters, $h = h(x,t)$ is another scalar field, $x \in \mathbb{T}^2$, and we write $\nabla^\perp = (-\partial_2, \partial_1)$. We remark that the model can be written in the formally equivalent form

$$\partial_t h + \nabla \cdot (uh) = 0 \tag{3}$$

augmented by the u-h relationship (2b). This can be verified by direct computation. However, the results in this paper hinge on the material conservation of potential vorticity, so that we only use the vorticity formulation of the model.

Physically, this model describes the large scale motion of a layer of shallow water in a rotating frame of reference in the limit of small Rossby number and similarly small Burger number, where h represents an approximation to the layer depth. It was first derived by Salmon [18] in two special cases, the so-called L_1 equations and the large-scale semigeostrophic equations. Subsequent work [14] generalized Salmon's approach, effectively interpreting different choices of μ as approximate near-identity changes of variables. In fact, $\sigma = \varepsilon(\lambda + \frac{1}{2})$ and $\mu = \varepsilon\lambda$ are convenient abbreviations for dependence on the physical parameters, the Rossby number ε and the factor of proportionality of the coordinate distortion which, to leading order, is given by $\lambda - \frac{1}{2}$. Details of the construction and higher order models can be found in [14]; the case of a spatially varying Coriolis parameter is discussed in [17]. Here, we only remark that the generalized LSG equations are formally valid in the same asymptotic regime in which Hoskins' semigeostrophic equations [12] are valid. The models differ in the higher order terms which are kept or discarded. Both classic semigeostrophy and the generalized LSG equations are Hamiltonian models, albeit with a different Hamiltonian structure.

Mathematically, the generalized LSG equations are interesting for the following reason. When $\mu \neq 0$, a superficial count of orders of differentiation in (2) indicates that the potential vorticity inversion should gain one derivative in Sobolev space, hence, the model is expected to behave like the Euler equations of an ideal fluid in two dimensions. When $\mu = 0$, the inversion is expected to gain three derivatives in Sobolev space, which would put this model on par with the so-called Euler-α or Lagrangian averaged Euler equations which were, in their inviscid form, first derived by Holm *et al.* [7] on formal grounds and later justified under certain closure assumptions in [6, 8]. Here, on the other hand, the additional regularity arises from terms which appear as part of an asymptotic expansion at the order to which the model is formally valid. In other words, we have an Euler-α-like model which has an, at this point formal, first principles derivation from the two-dimensional shallow water equations without the need for closure conditions or averaging.

It is not immediately apparent how to make the analogy with the Euler or Euler-α equations rigorous—there are notable differences to the standard global existence theory for two-dimensional inviscid flows. To appreciate the difficulties which arise, let us recall the essence of the classic argument. Global existence of H^s class classical solutions ultimately depends on a global estimate on the H^s norm of q. Thus, we compute

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} \|q\|_{H^s}^2 &\leq \left| \int \nabla \cdot u \, |\mathrm{D}^s q|^2 \, \mathrm{d}x \right| + \text{additional terms} \\ &\leq \|\nabla u\|_{L^\infty} \, \|q\|_{H^s}^2 + \text{additional terms} \end{aligned} \tag{4}$$

Note that we do not assume that u is divergence free; in our model, it is not. The "additional terms" which appear on the right of (4) can be shown to satisfy at worst an upper bound of the form already stated. To obtain a global H^s bound on q, we would need an estimate of $\|\nabla u\|_{L^\infty}$ in terms of $\|q\|_{H^s}$ which makes this differential inequality globally integrable. For example, when K is the vorticity inversion operator arising from the linear Biot–Savart law, it is known that

$$\|Kq\|_{W^{1,\infty}} \leq c \, \|q\|_{L^\infty} \left(1 + \ln_+ \|q\|_{H^s}\right) \tag{5}$$

in two and three dimensions with $s = 2$, implying global classical solutions for the two-dimensional Euler equations and the Beale–Kato–Majda criterion [2] for potential blowup of solutions for the three dimensional Euler equations.

A direct proof of (5) is achieved by careful estimates on the explicit Green's kernel of K. When an explicit formula for the kernel is not available, (5) can still be derived via L^p estimates for finite p provided the p-dependence of the estimate is essentially of the form

$$\|u\|_{W^{1,p}} \leq c \, \frac{p^2}{p-1} \, \|q\|_{L^p} \tag{6}$$

for $p \in (1, \infty)$. For ideal fluids, an estimate of this form follows from the general L^p theory of linear second order elliptic equations [20]. Then (5) follows by an optimal Gagliardo–Nirenberg estimate—this is the essence of the proof of our main Theorem 3. The idea behind this argument goes back to the work of Yudovich [21].

Implementing this strategy for the generalized LSG equations is nontrivial, however, as our potential vorticity inversion is nonlinear: even though (2a) and (2b) are linear in the unknown function, their coefficients depend on the data, so that their dependence on q and h, respectively, is nonlinear. Thus, the non-standard part of our argument consists of proving that the problem is not "too nonlinear" in the following sense.

First, an estimate of the form (6) holds true for the generalized LSG potential vorticity inversion with, however, a q-dependent constant c. In other words, its right hand side is now nonlinear in the potential vorticity, but we shall observe that this nonlinear dependence involves only upper and lower bounds on q which are constants of the motion. Second, when translating this estimate up the Sobolev scale as is necessary for estimating the "additional terms" in (4), all nonlinear dependence remains a function of constants of the motion. Third, in the higher rung translates of this estimate, the dependence on p becomes superlinear as $p \to \infty$; fortunately though, a careful look at the problem reveals that we can avoid taking this limit for all but the lowest rung estimate where the constant is indeed asymptotically linear in p.

We remark that a general theory for *linear* potential vorticity operators was formulated in [15], which also contains a review of the classic theory for ideal fluids in two dimensions. Further, in [16] we derive the Hamiltonian formulation of the generalized LSG equations and prove existence of local classical solutions. The proof given there is short and simple as it uses the topological algebra property of H^s for $s > 1$, but requires physically unreasonable smallness assumptions on the data and yields *a priori* bounds that blow up in a finite time. Here, for the first time, we have proof that Salmon's L_1 model and its generalizations, under the condition that the initial potential vorticity is positive, possess existence and regularity results as strong as those available for two dimensional ideal fluid flow. More precisely, the result, which is stated as Theorem 3 in the final section of this paper, is this: *Suppose $m \geq 3$ and the initial potential vorticity $q^{\text{in}} \in H^m(\mathbb{T}^2)$ is strictly positive. Then the generalized LSG equations* (1) *possess a global classical solution $q \in C^k([0,\infty); H^{m-k}(\mathbb{T}^2))$ for every $k = 0, \dots, m-2$. The associated velocity field satisfies $u \in C^k([0,\infty); H^{m-k+1}(\mathbb{T}^2))$ when $\mu \neq 0$ and $C^k([0,\infty); H^{m-k+3}(\mathbb{T}^2))$ when $\mu = 0$ for $k = 0, \dots, m-2$.* The restriction to positive potential vorticities is consistent with the assumed physical scaling in the derivation of the balance models, hence, is expected and reasonable. The generalized LSG equations also support weak solutions, but only under additional restrictions on the initial potential vorticity [3].

The article is structured as follows. In the next section we introduce our notation and conventions. In Section 3 we prove the main kinematic estimates which characterize our nonlinear potential vorticity operator K; the results are summarized in Theorem 1. Section 4 proves short-time existence and uniqueness of classical solutions for the generalized LSG equations. Finally, in Section 5 we extend the local classical solutions globally in time.

2. Notation and preliminaries

For $f \in L^\infty(\mathbb{T}^2)$, we define

$$f_- = \operatorname*{ess\,inf}_{x \in \mathbb{T}^2} f(x) \qquad \text{and} \qquad f_+ = \operatorname*{ess\,sup}_{x \in \mathbb{T}^2} f(x) \,. \tag{7}$$

Throughout this article, we assume that $0 < \sigma \leq 1$. The upper bound is nonessential. We carry out kinematic estimates assuming the potential vorticity is continuous with $q > 0$, so that $q \in [q_-, q_+]$ with q_- and q_+ being finite and positive. Furthermore, in the class of solutions we are considering, q_- and q_+ are constants of motion due to the advection of potential vorticity. Therefore, without loss of generality, we can assume that $q_+ - 1 = 1 - q_-$ and write $q \equiv 1 + \tilde{q}$ with $\|\tilde{q}\|_{L^\infty} < 1$. To see this, define in the general case $Q = (q_- + q_+)/2$, replace q by q/Q, h by hQ, ε by ε/Q, u by uQ, and t by t/Q, and note that the generalized LSG equations are invariant under this rescaling.

For $m \in \mathbb{N}_0$ and $1 \leq p \leq \infty$, we write $W^{m,p}(\mathbb{T}^2)$ to denote the Sobolev space of Lebesgue measurable functions whose weak derivatives up to order m belong to $L^p(\mathbb{T}^2)$, endowed with the norm

$$\|f\|_{W^{m,p}} = \sum_{|\alpha| \leq m} \|\mathrm{D}^\alpha f\|_{L^p} \,, \tag{8}$$

where we employ the usual multi-index notation. We abbreviate $H^m = W^{m,2}$; it is a Hilbert space with inner product

$$\langle f, g \rangle_m = \sum_{|\alpha| \leq m} \langle \mathrm{D}^\alpha f, \mathrm{D}^\alpha g \rangle_{L^2} \,. \tag{9}$$

In one instance, we need to refer to the space H^s with a non-integer exponent. In this case, we take, for convenience, the equivalent Fourier characterization where

$$\|f\|_{H^s}^2 = \sum_{k \in \mathbb{Z}^2} (1 + |k|^2)^s \, |\hat{f}_k|^2 \tag{10}$$

with $\hat{f}_k$ denoting the Fourier coefficients of f.

The space $W^{m,p}(\mathbb{T}^2)$ is a topological algebra when $mp > 2$. I.e., there exists a constant $c = c(m,p)$ such that

$$\|wv\|_{W^{m,p}} \leq c \, \|w\|_{W^{m,p}} \|v\|_{W^{m,p}} \tag{11}$$

for all $w, v \in W^{m,p}$. In particular, H^s is a topological algebra for $s > 1$.

For $m \in \mathbb{N}$ and $p \in (1, \infty)$ with Hölder conjugate p', we set $W^{-m,p}(\mathbb{T}^2) = (W^{m,p'}(\mathbb{T}^2))'$, endowed with the dual norm

$$\|f\|_{W^{-m,p}} = \sup_{\substack{\phi \in W^{m,p'} \\ \phi \neq 0}} \frac{\langle \phi, f \rangle}{\|\phi\|_{W^{m,p'}}} \,. \tag{12}$$

We remark that this definition coincides with the usual definition of $W^{-m,p}$ as the dual of $W_0^{m,p'}$ (see, e.g., [1]), because on the torus the spaces $W^{m,p'}$ and $W_0^{m,p'}$ coincide.

Finally, we define potential vorticity "balls," on which the estimates we will derive are uniform, as

$$A^{m,p}(r, R) = \{\tilde{q} \in W^{m,p}(\mathbb{T}^2) \colon \|\tilde{q}\|_{L^\infty} < r \text{ and } \|\tilde{q}\|_{W^{m,p}} < R\} \tag{13}$$

for $R > 0$ and $0 \leq r < 1$; we abbreviate $A^m \equiv A^{m,2}$.

We adopt the following convention on the naming of constants. Constants that might depend on parameters only are denoted by c; constants that may also depend on the data or the bounds r and R of the vorticity balls are denoted C. Different subscripts indicate a change in the constant from step to step within a single computation; however, we make no attempt at a unique naming of constants across different sections of the paper. Nonuniform dependence on $p \in (1,\infty)$ is always indicated explicitly; nonuniform dependence on σ, where, for simplicity, we always assume $\sigma \leq 1$, is only tracked in Section 3.

3. Kinematic estimates

In this section we establish sufficient conditions under which the operator K is well defined and derive kinematic estimates for later use. This task naturally splits in three parts: we first study the second order differential operator from (2a), which we abbreviate

$$L_q = q - \sigma \varDelta \,. \tag{14}$$

Second, we look at the second order differential operator on the left hand side of (2b), which we abbreviate

$$\varLambda_h = 1 - \sigma \, (h \, \varDelta + 2 \, \nabla h \cdot \nabla) \tag{15}$$

supposing that h is already given as a solution of (2a). Finally, we include the right hand side of (2b), thereby completing the estimates for the full potential vorticity inversion. The final result is stated as Theorem 1 toward the end of this section.

Proposition 1. *Suppose $f \in L^p(\mathbb{T}^2)$ with $1 < p < \infty$ and $\tilde{q} \in L^\infty(\mathbb{T}^2)$ with $\|\tilde{q}\|_{L^\infty} \leq r < 1$. Then the equation*

$$L_q h = f \tag{16}$$

has a unique solution $h \in W^{2,p}$ with

$$\|h\|_{L^p} \leq \frac{1}{1-r} \, \|f\|_{L^p} \tag{17a}$$

and, for some $c > 0$,

$$\|h\|_{W^{2,p}} \leq \frac{c}{\sigma} \, \frac{p^2}{p-1} \, \frac{1}{1-r} \, \|f\|_{L^p} \,. \tag{17b}$$

If, in addition, $f, q \in W^{m,p}(\mathbb{T}^2)$ with $m \in \mathbb{N}$, then $h \in W^{m+2,p}$ and there exists a constant C depending on r and on all parameters such that

$$\|h\|_{W^{m+2,p}} \leq C \left(1 + \|\tilde{q}\|_{W^{m,p}}\right) \|f\|_{W^{m,p}} \,. \tag{18}$$

In both cases above, the map $q \mapsto L_q^{-1} f$ is uniformly continuous on $\tilde q \in A^{m,p}(r,R)$ for every $r \in [0,1)$ and $R > 0$ as a map from $W^{m,p}$ to $W^{m+2,p}$. Specifically, there exists a constant C depending on all parameters as well as r, R, and $\|f\|_{W^{m,p}}$ such that

$$\|L_{q_2}^{-1} f - L_{q_1}^{-1} f\|_{W^{m+2,p}} \leq C \, \|q_2 - q_1\|_{W^{m,p}} \tag{19}$$

for all $\tilde q_1, \tilde q_2 \in A^{m,p}(r,R)$.

Proof. Since $\|\tilde q\|_{L^\infty} < 1$, the second order operator L_q is uniformly elliptic and coercive. Hence, existence and uniqueness of a solution in $W^{2,p}$ follow directly from standard elliptic L^p theory. To proceed, we write (16) in fixed point form, namely,

$$h = (1 - \sigma \Delta)^{-1} (f - \tilde q \, h) \,. \tag{20}$$

Hence,

$$\|h\|_{L^p} \leq \|(1 - \sigma\Delta)^{-1}\|_{L^p \to L^p} \left(\|f\|_{L^p} + \|\tilde q\|_{L^\infty} \, \|h\|_{L^p} \right) . \tag{21}$$

Since the inverse Helmholtz operator has unit norm on L^p, as can be seen from its explicit integral representation, estimate (17a) follows immediately. Moreover, elliptic L^p regularity [20, 4, 10] provides that there exists a constant c such that

$$\|(1 - \sigma\Delta)^{-1}\|_{L^p \to W^{2,p}} \leq \frac{c}{\sigma} \, \frac{p^2}{p-1} \,. \tag{22}$$

Hence, considering the inverse Helmholtz operator as a map from L^p to $W^{2,p}$ in (21) yields (17b).

Higher regularity with corresponding estimates in $W^{m,p}$ is proved by an induction argument: suppose the statement is already proved up to some integer $m \geq 0$ and let α be a multi-index with $|\alpha| = m+1$. Then

$$\mathrm{D}^\alpha h = (1 - \sigma\Delta)^{-1} \left(\mathrm{D}^\alpha f - \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} \mathrm{D}^{\alpha-\beta} \tilde q \, \mathrm{D}^\beta h \right) . \tag{23}$$

To estimate the L^p norm of each of the terms in the right hand sum, we apply Hölder inequalities with conjugate exponents

$$s = \frac{|\alpha|}{|\alpha - \beta|} \quad \text{and} \quad s' = \frac{|\alpha|}{|\beta|} \,, \tag{24}$$

use the induction assumption (in combination with (17b) and the Sobolev embedding $W^{2,p}(\mathbb{T}^2) \subset L^\infty(\mathbb{T}^2)$ in the case $\beta = 0$), and apply Gagliardo–Nirenberg inequalities [9, 13] with dimension independent exponents on each norm of $\tilde q$ and a Sobolev embedding on the norm of f, to obtain

$$\begin{aligned}
\|\mathrm{D}^{\alpha-\beta} \tilde q \, \mathrm{D}^\beta h\|_{L^p} &\leq \|\mathrm{D}^{\alpha-\beta} \tilde q\|_{L^{ps}} \, \|\mathrm{D}^\beta h\|_{L^{ps'}} \\
&\leq \|\mathrm{D}^{\alpha-\beta} \tilde q\|_{L^{ps}} \, C_1 \left(1 + \|\tilde q\|_{W^{|\beta|,ps'}}\right) \|f\|_{W^{|\beta|,ps'}} \\
&\leq C_2 \, \|\mathrm{D}^{|\alpha|} \tilde q\|_{L^p}^{\frac{1}{s}} \, \|\tilde q\|_{L^\infty}^{1-\frac{1}{s}} \left[1 + \|\tilde q\|_{W^{|\alpha|,p}}^{\frac{1}{s'}} \, \|\tilde q\|_{L^\infty}^{1-\frac{1}{s'}} \right] \|f\|_{W^{|\alpha|,p}} \\
&\leq C_3 \left(1 + \|\tilde q\|_{W^{|\alpha|,p}}\right) \|f\|_{W^{|\alpha|,p}} \,.
\end{aligned} \tag{25}$$

We remark that the p-dependence of the Gagliardo–Nirenberg constants is nontrivial. It could be tracked down via integration by parts and Hölder inequalities; the details do not matter for our purposes when $m \geq 1$.

Inserting (25) into a $W^{2,p}$ estimate of (23), we find that

$$\begin{aligned} \|\mathrm{D}^\alpha h\|_{W^{2,p}} \leq \|(1-\sigma\Delta)^{-1}\|_{L^p\to W^{2,p}} \\ \cdot \left[\|\mathrm{D}^\alpha f\|_{L^p} + C_4\left(1+\|\tilde{q}\|_{W^{|\alpha|,p}}\right)\|f\|_{W^{|\alpha|,p}}\right]. \end{aligned} \tag{26}$$

Estimate (18) is an immediate consequence.

Finally, to prove uniform continuity, suppose under the respective assumptions on m and p that $\tilde{q}_i \in A^{m,p}(r,R)$ and $L_{q_i} h_i = f$ for $i = 1, 2$. Then

$$h_2 - h_1 = L_{q_2}^{-1}[h_1\,(q_1 - q_2)] \tag{27}$$

so that, due to (17b) or (18), respectively,

$$\|h_2 - h_1\|_{W^{m+2,p}} \leq C(p,\sigma,r,R)\,\|h_1\,(q_2-q_1)\|_{W^{m,p}}\,. \tag{28}$$

The claim then follows by noting that, in two dimensions, the norm on the right can always be estimated by the product of $\|q_2 - q_1\|_{W^{m,p}}$ and $\|h_1\|_{W^{m+2,p}}$. □

Remark 1. When $f \in L^\infty(\mathbb{T}^2) \cap W^{m,p}(\mathbb{T}^2)$, a similar induction argument, where the term where $\alpha = \beta$ in the Leibniz expansion is moved onto the left hand side of the estimate, yields a bound with optimal dependence in σ which is, moreover, linear in derivatives of the data. Namely, there is a constant C which depends only on m, p, and $\|\tilde{q}\|_{L^\infty}$ such that

$$\|h\|_{W^{m,p}} \leq C\left(\|f\|_{W^{m,p}} + \|f\|_{L^\infty}\,\|\tilde{q}\|_{W^{m,p}}\right) \tag{29}$$

and

$$\|h\|_{W^{m+2,p}} \leq \frac{C}{\sigma}\left(\|f\|_{W^{m,p}} + \|f\|_{L^\infty}\,\|\tilde{q}\|_{W^{m,p}}\right) \tag{30}$$

For $m = 1$, the condition that $f \in L^\infty$ may be dropped with $\|f\|_{L^\infty}$ replaced by $\|h\|_{L^\infty}$.

Proposition 2. *Suppose $\tilde{q} \in L^\infty(\mathbb{T}^2)$ with $\|\tilde{q}\|_{L^\infty} < 1$ and let h be the solution to $L_q h = 1$ given by Proposition 1. Then*

$$\frac{1}{q_+} \leq h \leq \frac{1}{q_-}\,. \tag{31}$$

Proof. We rewrite the equation $L_q h = 1$ in the form

$$L_q\left(h - \frac{1}{q_+}\right) = 1 - \frac{q}{q_+} \geq 0\,. \tag{32}$$

First, suppose that $q \in C(\mathbb{T}^2)$ and $h \in C^2(\mathbb{T}^2)$. Since L_q is uniformly elliptic, the classical strong maximum principle [11,10] then implies

$$h - \frac{1}{q_+} \geq 0 \,. \tag{33}$$

The upper bound on h follows from the corresponding argument for $h - 1/q_-$. The general case when $\tilde{q} \in L^\infty(\mathbb{T}^2)$ follows by a standard mollification argument. □

We now turn to studying the operator Λ_h defined in (15). We begin by stating a result on weak solutions of a linear second order equation in $W^{1,p}$ which is a direct consequence of the well-known L^p theory for elliptic operators [4,10,20].

Lemma 1. *Suppose $f \in W^{-1,p}(\mathbb{T}^2)$ with $2 \leq p < \infty$. Then the equation*

$$(1 - \sigma\Delta)v = f \tag{34}$$

has a unique weak solution $v \in W^{1,p}(\mathbb{T}^2)$ and there exists a constant c independent of p and σ such that

$$\|v\|_{W^{1,p}} \leq \frac{c\,p}{\sigma} \, \|f\|_{W^{-1,p}} \,. \tag{35}$$

Proof. Since $H^{-1} \supset W^{-1,p}$, existence of a unique weak solution $v \in H^1$ is elementary. To show that $v \in W^{1,p}$, it suffices to prove (35) for $f \in W^{1,p}$ and to by density. Indeed, due to (22),

$$\begin{aligned} \|v\|_{L^p} &= \sup_{\substack{\phi \in L^{p'} \\ \phi \neq 0}} \frac{\langle \phi, v \rangle_{L^2}}{\|\phi\|_{L^{p'}}} \leq \frac{c}{\sigma} \, \frac{(p')^2}{p'-1} \sup_{\substack{\psi \in W^{2,p'} \\ \psi \neq 0}} \frac{\langle (1-\sigma\Delta)\psi, v \rangle_{L^2}}{\|\psi\|_{W^{2,p'}}} \\ &\leq \frac{2\,c\,p}{\sigma} \, \|f\|_{W^{-2,p}} \,. \end{aligned} \tag{36}$$

Since $(1 - \sigma\Delta)\partial_i v = \partial_i f$ for $i = 1, 2$, this estimate also yields

$$\|\partial_i v\|_{L^p} \leq \frac{2\,c\,p}{\sigma} \, \|\partial_i f\|_{W^{-2,p}} \leq \frac{2\,c\,p}{\sigma} \, \|f\|_{W^{-1,p}} \,. \tag{37}$$

Estimates (36) and (37) finally imply (35). □

We now establish a corresponding result for weak solutions of $\Lambda_h u = g$. As usual, a weak solution is a function $u \in H^1(\mathbb{T}^2)$ which satisfies

$$B(u, v) = \langle g, v \rangle \tag{38}$$

for every $v \in H^1(\mathbb{T}^2)$, where the bilinear form B reads

$$B(u, v) = \int_{\mathbb{T}^2} \bigl(u \cdot v + \sigma\, h \, \nabla u : \nabla v - \sigma \, \nabla h \cdot (\nabla u)^T v \bigr) \, \mathrm{d}x \,, \tag{39}$$

the colon denoting summation of componentwise products over both indices. We can then prove the following.

Proposition 3. *Suppose $\tilde q \in L^\infty(\mathbb{T}^2)$ with $\|\tilde q\|_{L^\infty} \leq r < 1$ and let h be the solution to $L_q h = 1$ given by Proposition 1. Further, let $g \in W^{-1,p}(\mathbb{T}^2)$ with $2 \leq p < \infty$. Then the problem*

$$\Lambda_h u = g \tag{40}$$

has a unique weak solution $u \in W^{1,p}(\mathbb{T}^2)$ and there exists a constant c independent of p such that

$$\|u\|_{W^{1,p}} \leq \frac{c\,p}{\sigma^2} \, \frac{1}{1-r} \, \|g\|_{W^{-1,p}} \,. \tag{41}$$

In particular, when g denotes the right hand side of the generalized LSG momentum equation (2b)*, then there exists a constant C_1 independent of p but dependent on all other parameters as well as on r such that*

$$\|u\|_{W^{1,p}} \leq C_1 \, p \,. \tag{42}$$

Finally, when $g \in L^p(\mathbb{T}^2)$ for any $p \in (1,\infty)$, there exists a constant C_2 depending on p, r, and on all other parameters such that

$$\|u\|_{W^{2,p}} \leq C_2 \, \|g\|_{L^p} \,. \tag{43}$$

Remark 2. The prefactor $1/\sigma^2$ in (41) is not optimal. For $p = 2$, our proof below shows that the constant scales like $1/\sigma$ for $\sigma \to 0$. In general, one could trade extra dependence of the constant in p for an improvement in the dependence on σ; this, however, is not relevant for the purposes of our work here.

Proof. We first assume $p \geq 2$ so that $g \in H^{-1}(\mathbb{T}^2)$. As in the proof of Lemma 1, we establish existence of a unique weak solution $u \in H^1(\mathbb{T}^2)$ by the Lax–Milgram theorem. Continuity of the bilinear form (39) is immediate. To prove coercivity, we write

$$\begin{aligned} B(u,u) &= \int_{\mathbb{T}^2} \bigl(|u|^2 + \sigma \, h \, |\nabla u|^2 - \tfrac12 \, \sigma \, \nabla h \cdot \nabla |u|^2 \bigr) \, \mathrm{d}x \\ &= \int_{\mathbb{T}^2} \bigl(|u|^2 + \sigma \, h \, |\nabla u|^2 + \tfrac12 \, \sigma \, \Delta h \, |u|^2 \bigr) \, \mathrm{d}x \\ &= \int_{\mathbb{T}^2} \bigl(\tfrac12 \, (1 + qh) \, |u|^2 + \sigma \, h \, |\nabla u|^2 \bigr) \, \mathrm{d}x \\ &\geq \min\{\tfrac12, \sigma h_-\} \, \|u\|_{H^1}^2 \,. \end{aligned} \tag{44}$$

Since, by Proposition 2, $h_- \geq 1/q_+ > 1/2$, and $\sigma \leq 1$ throughout, the Lax–Milgram theorem asserts existence of a unique weak solution $u \in H^1(\mathbb{T}^2)$ with

$$\|u\|_{H^1} \leq \frac{2}{\sigma} \, \|g\|_{H^{-1}} \,. \tag{45}$$

To prove $W^{1,p}$ regularity, we write $1/h = 1 + \tilde b$. We recall from Proposition 2 that $1 + \tilde q_- \leq 1/h \leq 1 + \tilde q_+$, so that

$$\|\tilde b\|_{L^\infty} \leq \|\tilde q\|_{L^\infty} \,. \tag{46}$$

Then (40) can be written as

$$(1-\sigma\Delta)u = \frac{g}{h} - \tilde{b}\,u + \frac{2\sigma}{h}\,\nabla h\cdot\nabla u\,. \tag{47}$$

So, to prove the claim, it suffices to find appropriate estimates in $W^{-1,p}$ for each of the terms on the right.

To begin, continuity of the embedding $W^{1,4}(\mathbb{T}^2)\subset L^\infty(\mathbb{T}^2)$ and estimate (17b) applied with fixed $p=4$ imply that there exists a constant c_1 independent of p such that

$$\|\nabla h\|_{L^\infty} \le \frac{c_1}{\sigma}\,\frac{1}{1-r}\,. \tag{48}$$

Using Proposition 2, we then find that for every $\phi\in W^{1,p'}(\mathbb{T}^2)$,

$$\|\phi/h\|_{W^{1,p'}} \le \frac{c_2}{\sigma}\,\frac{1}{1-r}\,\|\phi\|_{W^{1,p'}}\,. \tag{49}$$

Hence, when $g\in L^2(\mathbb{T}^2)$,

$$\|g/h\|_{W^{-1,p}} \le \frac{c_2}{\sigma}\,\frac{1}{1-r}\sup_{\substack{\phi\in W^{1,p'}\\ \phi\neq 0}}\frac{\langle\phi/h,g\rangle_{L^2}}{\|\phi/h\|_{W^{1,p'}}} = \frac{c_2}{\sigma}\,\frac{1}{1-r}\,\|g\|_{W^{-1,p}}\,. \tag{50}$$

This inequality extends to all $g\in W^{-1,p}(\mathbb{T}^2)$ by density. Next,

$$\|\tilde{b}u\|_{W^{-1,p}} \le c_3\,\|\tilde{b}u\|_{L^2} \le \frac{c_4}{\sigma}\,\|\tilde{q}\|_{L^\infty}\,\|g\|_{W^{-1,p}}\,, \tag{51}$$

where the second inequality is due to the Sobolev embedding theorem, (45), and (46). Using (48), in addition, we finally obtain

$$\|h^{-1}\,\nabla h\cdot\nabla u\|_{W^{-1,p}} \le c_3\,\|h^{-1}\|_{L^\infty}\,\|\nabla h\|_{L^\infty}\,\|\nabla u\|_{L^2} \le \frac{c_5}{\sigma^2}\,\frac{1}{1-r}\,\|g\|_{W^{-1,p}}\,. \tag{52}$$

This concludes the proof of (41).

To prove (42), we note that when g is given by the right hand side of (2b),

$$\|g\|_{W^{-1,p}} \le \|h-\mu\,(2\,h\,\Delta h + |\nabla h|^2)\|_{L^p}\,. \tag{53}$$

Since (2a) implies $\Delta h=(qh-1)/\sigma$ and h is bounded in L^∞ due to Proposition 2, the first two terms inside the L^p norm in (53) are bounded in L^∞, hence in all L^p uniformly in p. The last term is bounded in L^∞ due to (48). The claim then follows from (41).

Finally, to prove (43), it suffices to obtain an appropriate L^p bound on the expression on the right of (47). We estimate

$$\|h^{-1}g\|_{L^p} \le \|h^{-1}\|_{L^\infty}\,\|g\|_{L^p}\,, \tag{54}$$

$$\|\tilde{b}\,u\|_{L^p} \le c_6\,\|\tilde{q}\|_{L^\infty}\,\|u\|_{L^{2p}} \le C_1\,\|g\|_{W^{-1,2p}} \le C_2\,\|g\|_{L^p}\,, \tag{55}$$

and

$$\|h^{-1}\nabla h \cdot \nabla u\|_{L^p} \leq \|h^{-1}\|_{L^\infty} \|h\|_{W^{1,2p}} \|u\|_{W^{1,2p}} \\ \leq C_3 \|g\|_{W^{-1,2p}} \leq C_4 \|g\|_{L^p} , \tag{56}$$

where the second inequality is due to Proposition 1 and estimate (41). □

Having established existence and uniqueness of a weak solution of (2b), we expect, in analogy with linear elliptic regularity theory, the existence of a solution $u \in W^{m+1,p}(\mathbb{T}^2)$ when $q \in W^{m,p}(\mathbb{T}^2)$. In Theorem 1 we show that this is indeed the case and that, although the dependence of u on q is nonlinear, the bounds on $\|u\|_{W^{m+1,p}}$ are linear in derivatives of q. We start by stating a simple fact about the reciprocals of $W^{m,p}$ functions.

Lemma 2. *Suppose $h \in W^{m,p}(\mathbb{T}^2) \cap L^\infty(\mathbb{T}^2)$ for $m \in \mathbb{N}$ and $p \in (1,\infty)$ with $h_- > 0$. Then $h^{-1} \in W^{m,p}(\mathbb{T}^2)$ and there exists a constant $C = C(m,p,h_-,h_+)$ such that*

$$\|h^{-1}\|_{W^{m,p}} \leq C \|h\|_{W^{m,p}} . \tag{57}$$

Proof. By the multivariate Faà di Bruno formula [5], the terms in a full expansion of $\mathrm{D}^\alpha h^{-1}$ are all of the form

$$h^{-\ell-1} \, \mathrm{D}^{\alpha^1} h \cdots \mathrm{D}^{\alpha^\ell} h , \tag{58}$$

where $\alpha^1 + \cdots + \alpha^\ell = \alpha$ is a partition of the multi-index α into ℓ parts. We estimate

$$\|h^{-\ell-1} \, \mathrm{D}^{\alpha^1} h \cdots \mathrm{D}^{\alpha^\ell} h\|_{L^p} \leq \|h^{-1}\|_{L^\infty}^{\ell+1} \prod_{i=1}^{\ell} \|\mathrm{D}^{\alpha^i} h\|_{L^{ps_i}} \\ \leq c \, \|h^{-1}\|_{L^\infty}^{\ell+1} \prod_{i=1}^{\ell} \|\mathrm{D}^{|\alpha|} h\|_{L^p}^{\frac{1}{s_i}} \|h\|_{L^\infty}^{1-\frac{1}{s_i}} \\ \leq C \, \|\mathrm{D}^{|\alpha|} h\|_{L^p} , \tag{59}$$

where the first inequality is due to the Hölder inequality with $s_i = |\alpha|/|\alpha^i|$ and the second inequality employs the dimension independent case of the Gagliardo–Nirenberg inequality. The claim follows directly. □

The properties of the full potential vorticity inversion are now stated in the following theorem.

Theorem 1. *Suppose $\tilde{q} \in W^{m,p}(\mathbb{T}^2) \cap L^\infty(\mathbb{T}^2)$, $m \in \mathbb{N}_0$, $1 < p < \infty$, and $(m+1)p \geq 2$ with $\|\tilde{q}\|_{L^\infty} \leq r < 1$. Let $h = h(q)$ denote the solution to $L_q h = 1$ given by Proposition 1, let $g = g(q)$ denote the right hand side of the generalized LSG momentum equation* (2b)*, and let $u \equiv K(q)$ denote the solution to $\Lambda_h u = g$ given by Proposition 3. Then $u \in W^{m+1,p}$ and there exists a constant C depending on r and on all parameters such that*

$$\|u\|_{W^{m+1,p}} \leq C \|q\|_{W^{m,p}} . \tag{60}$$

In the special case when $\mu = 0$*,* $u \in W^{m+3,p}$ *with bound*

$$\|u\|_{W^{m+3,p}} \leq C \, \|q\|_{W^{m,p}} \,. \tag{61}$$

Finally, for $m > 0$ *and* $R > 0$ *fixed, the operator* K *is uniformly continuous on the set* $\tilde{q} \in A^{m,p}(r,R)$ *as a map from* $W^{m,p}$ *into* $W^{m+1,p}$ *in general and from* $W^{m,p}$ *into* $W^{m+3,p}$ *when* $\mu = 0$*.*

Proof. Estimate (18) of Proposition 1 already provides the bound

$$\|h\|_{W^{m+2,p}} \leq C_1 \, \|q\|_{W^{m,p}} \tag{62}$$

on the solution of $L_q h = 1$. So (60) follows provided we can show that there exists a constant C_2, depending on r and on all parameters, such that

$$\|u\|_{W^{m+1,p}} \leq C_2 \, \|h\|_{W^{m+2,p}} \,. \tag{63}$$

When $m = 0$ with $2 \leq p < \infty$, estimate (63) follows directly from estimate (42) of Proposition 3. When $m = 1$ with $1 < p < \infty$, (63) follows from (43) of Proposition 3, where we need a bound on $\|g\|_{L^p}$. This bound is achieved by noting that, for arbitrary multi-indexes α and β, dimension independent Gagliardo–Nirenberg inequalities yield

$$\|\mathrm{D}^\alpha h \, \mathrm{D}^\beta h\|_{L^p} \leq c \, \|h\|_{W^{|\alpha+\beta|,p}} \, \|h\|_{L^\infty} \,. \tag{64}$$

The proof is identical to the proof of Lemma 2. Combining (62) with (64) and (31) we then obtain for any $m \in \mathbb{N}_0$ that

$$\|g(q)\|_{W^{m-1,p}} \leq C_3 \, \|q\|_{W^{m,p}} . \tag{65}$$

In the general case, we proceed by induction. Suppose the statement is already proved up to some $m \in \mathbb{N}$ and let α be a multi-index with $|\alpha| = m$. Differentiating (47), we obtain

$$(1 - \sigma \mathit{\Delta}) \mathrm{D}^\alpha u = \mathrm{D}^\alpha (h^{-1} g) - \mathrm{D}^\alpha (\tilde{b} \, u) + 2 \, \sigma \, \mathrm{D}^\alpha (h^{-1} \, \nabla h \cdot \nabla u) \,. \tag{66}$$

Noting that $(1 - \sigma \mathit{\Delta})^{-1}$ is continuous from L^p to $W^{2,p}$, we obtain the required bound on $\|u\|_{W^{m+2,p}}$ via an L^p estimate on the right hand side. We begin with the last term, estimating

$$\begin{aligned}
&\|\mathrm{D}^\alpha (h^{-1} \, \nabla h \cdot \nabla u)\|_{L^p} \\
&\quad \leq c \sum_{\alpha^1 + \alpha^2 + \alpha^3 = \alpha} \|\mathrm{D}^{\alpha^1} h^{-1}\|_{L^{ps_1}} \, \|\mathrm{D}^{\alpha^2} \nabla h\|_{L^{ps_2}} \, \|\mathrm{D}^{\alpha^3} \nabla u\|_{L^{ps_3}} \\
&\quad \leq C_4 \sum_{\alpha^1 + \alpha^2 + \alpha^3 = \alpha} \|h\|_{W^{|\alpha^1|,ps_1}} \, \|h\|_{W^{|\alpha^2|+1,ps_2}} \, \|h\|_{W^{|\alpha^3|+2,ps_3}} \\
&\quad \leq C_5 \, \|h\|_{W^{|\alpha|+3,p}} \, \|h\|_{L^\infty}^2 \,,
\end{aligned} \tag{67}$$

where the first inequality is due to Hölder inequalities with

$$s_1 = \frac{|\alpha| + 3}{|\alpha^1|} \,, \qquad s_2 = \frac{|\alpha| + 3}{|\alpha^2| + 1} \,, \qquad \text{and} \qquad s_3 = \frac{|\alpha| + 3}{|\alpha^3| + 2} \,, \tag{68}$$

the second inequality in (67) is due to Lemma 2 and the induction hypothesis, and the final inequality in (67) is due to Gagliardo–Nirenberg inequalities with dimension independent exponents where we note that, as in the proof of Lemma 2, the exponents always add up to one.

The second term on the right of (66) is of lower order relative to the third so that it enjoys the same type of upper bound. For the first term, we insert the explicit form of g from (2b). The term with the highest order derivative reads $\|\mathrm{D}^\alpha \Delta \nabla h\|_{L^p}$ and is, therefore, already of the required form. The expression further contains multiple terms with the same homogeneity as those on the second line of (67), as well as lower order contributions which are easily seen to satisfy the required upper bound. Altogether, this completes the proof of (63), hence of (60).

Let us now turn to the special case when $\mu = 0$ so that $g = \nabla^\perp h$. Here, it suffices to show that for every $m \geq 2$ there exists a constant C_1, which may depend on r and on the remaining parameters, such that

$$\|u\|_{W^{m+1,p}} \leq C_1 \, \|h\|_{W^{m,p}} \, . \tag{69}$$

Due to (62), estimate (61) is an immediate consequence.

To prove (69), we first note that due to estimate (43) of Proposition 3 and the boundedness of h in $W^{1,\infty}$, for every $s \in (1, \infty)$ there exists a constant C_2 depending on s, r, and on the remaining parameters such that

$$\|u\|_{W^{2,s}} \leq C_2 \, . \tag{70}$$

We now proceed inductively. Suppose (69) is already proved up to order $m' \in \mathbb{N}$, and let α be a multi-index with $|\alpha| = m'$. Hence, to prove (69) for $m = m' + 1$, we need to estimate the right hand side of (66) in L^p. Here, $g = \nabla^\perp h$ so that the first and second terms are of lower order relative to the third term, and shall not be considered explicitly. To obtain a bound on the third term, we proceed as in (67), but with exponents

$$s_1 = \frac{|\alpha| + 1}{|\alpha^1|} \, , \qquad s_2 = \frac{|\alpha| + 1}{|\alpha^2| + 1} \, , \qquad \text{and} \qquad s_3 = \frac{|\alpha| + 1}{|\alpha^3|} \, . \tag{71}$$

Employing Lemma 2 and using the induction hypothesis when $|\alpha^3| \geq 2$, (70) when $|\alpha^3| = 1$, and (70) in combination with a Sobolev embedding when $|\alpha^3| = 0$, we find

$$\begin{aligned} &\|\mathrm{D}^\alpha (h^{-1} \, \nabla h \cdot \nabla u)\|_{L^p} \\ &\quad \leq C_3 \sum_{\alpha^1 + \alpha^2 + \alpha^3 = \alpha} \|h\|_{W^{|\alpha^1|, ps_1}} \, \|h\|_{W^{|\alpha^2|+1, ps_2}} \, \|h\|_{W^{|\alpha^3|, ps_3}} \\ &\quad \leq C_4 \, \|h\|_{W^{|\alpha|+1,p}} \, \|h\|_{L^\infty}^2 \, , \end{aligned} \tag{72}$$

where, once again, we used dimension independent Gagliardo–Nirenberg inequalities in the final step. This concludes the proof of (69), hence, of (61).

It remains to verify the claim of uniform continuity on $A^{m,p}(r,R)$. We note that, due to (2a),

$$\Lambda_h u = L_q(hu)\,. \tag{73}$$

Suppose $\tilde q_1, \tilde q_2 \in A^{m,p}(r,R)$ and set $h_i = h(q_i)$ and $u_i = K(q_i)$. Using (73), we obtain

$$\begin{aligned} u_1 - u_2 &= \frac{1}{h_1 h_2}\left[h_2 L_{q_1}^{-1}(g(q_1) - h_1 L_{q_2}^{-1}(g(q_2))\right] \\ &= \frac{h_2 - h_1}{h_1 h_2}\, L_{q_1}^{-1} g(q_1) + \frac{L_{q_1}^{-1} g(q_1) - L_{q_2}^{-1} g(q_1)}{h_2} + \frac{L_{q_2}^{-1}(g(q_1) - g(q_2))}{h_2} \end{aligned} \tag{74}$$

and note that we have uniform bounds on the norms $\|h_i\|_{W^{m+2,p}}$ due to (18), on $\|h_i^{-1}\|_{W^{m+2,p}}$ due to Lemma 2, on $\|g(q_i)\|_{W^{m-1,p}}$ due to (65), and on $\|L_{q_j}^{-1} g(q_i)\|_{W^{m+1,p}}$ due to Proposition 1. Further, we have

$$\begin{aligned} \|g(q_1) - g(q_2)\|_{W^{m-1,p}} &\leq \|h_1 - h_2\|_{W^{m,p}} + 2\mu\, \|h_1\, \Delta h_1 - h_2\, \Delta h_2\|_{W^{m,p}} \\ &\quad + \mu\, \|\nabla(h_1 - h_2)\cdot\nabla(h_1 + h_2)\|_{W^{m,p}} \\ &\leq C\,\|q_1 - q_2\|_{W^{m,p}}\,. \end{aligned} \tag{75}$$

We then obtain uniform continuity of K as an operator from $W^{m,p}$ to $W^{m+1,p}$ by taking the $W^{m+1,p}$-norm of (74), noting that $W^{m+1,p}(\mathbb{T}^2)$ is a topological algebra when $m \geq 1$, and using (75) in combination with (19).

When $\mu = 0$, the corresponding estimate on (74) gives uniform continuity of K as an operator from $W^{m,p}$ to $W^{m+2,p}$; this can be used to achieve uniform continuity into $W^{m+3,p}$ via (66) with $|\alpha| = m+1$, noting the multilinearity of its right hand side. □

4. Local classical solutions

Theorem 2. *Let $m \geq 3$, $0 \leq r < 1$, and $R > 0$. Then there exists a time $T = T(m,r,R)$ such that for every initial potential vorticity anomaly $\tilde q^{\text{in}} \in A^m(r,R)$ there exists a unique classical solution to the generalized LSG equations* (1) *of class*

$$q \in \bigcap_{k=0}^{m-2} C^k([0,T); H^{m-k}(\mathbb{T}^2))\,, \tag{76a}$$

$$h \in \bigcap_{k=0}^{m-2} C^k([0,T); H^{m+2-k}(\mathbb{T}^2))\,, \tag{76b}$$

$$u \in \bigcap_{k=0}^{m-2} C^k([0,T); H^{m+1-k}(\mathbb{T}^2))\,. \tag{76c}$$

Moreover, when $\mu = 0$,

$$u \in \bigcap_{k=0}^{m-2} C^k([0,T); H^{m+3-k}(\mathbb{T}^2)) \,. \tag{77}$$

Proof. We construct the solution as the limit of a Galerkin approximating sequence as in [15, 19]. We proceed in several steps.

Step 1. *Construct a family of approximate solutions* $\{q_n\}$.

Let P_n denote the H^m-orthogonal projector onto the Fourier modes with wave number less or equal to n in modulus, and set $Q_n = 1 - P_n$. We consider the vorticity equation (1a) restricted to $P_n H^m(\mathbb{T}^2)$,

$$\partial_t q_n + P_n(u_n \cdot \nabla q_n) = 0 \,, \tag{78a}$$
$$u_n = K(q_n) \,, \tag{78b}$$
$$q_n(0) = q_n^{\text{in}} \equiv P_n q^{\text{in}} \,. \tag{78c}$$

As before, we write $q_n = 1 + \tilde{q}_n$. Since the inclusion $H^m(\mathbb{T}^2) \hookrightarrow C(\mathbb{T}^2)$ is continuous, the convergence $q_n^{\text{in}} \to q^{\text{in}}$ in H^m implies that $\tilde{q}_n^{\text{in}} \in A^m(\frac{r+1}{2}, R)$ for $n \geq N$ with N sufficiently large. Then, by the Picard–Lindelöf theorem for ordinary differential equations, there exists a time T_n such that (78) possesses a solution with $\tilde{q}_n(t) \in A^m(\frac{r+3}{4}, R)$ for $t \in [0, T_n)$.

Step 2. *Show that* $\liminf_{n\to\infty} T_n > 0$.

Proof. We first note that, unlike in the case of ideal planar flow where K denotes the two-dimensional Biot–Savart operator, it is not sufficient to obtain uniform control on the H^m norm of q_n. Here, solutions may also cease to exist once we violate the solvability condition $\|\tilde{q}_n\|_{L^\infty} < 1$. We argue as follows. Let

$$T_n = \sup\{T : \tilde{q}_n(t) \in A^m(\tfrac{r+3}{4}, R) \text{ for } t \in [0, T)\} \,. \tag{79}$$

Without loss of generality we may assume that $T_n \leq \hat{T} < \infty$ for all $n \in \mathbb{N}$. Noting that $\tilde{q}_n$ also satisfies an equation of the form (78a), multiplying by $\tilde{q}_n^{p-1}$, and integrating over the torus, we obtain

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} \|\tilde{q}_n(t)\|_{L^p} &\leq \|\tilde{q}_n\|_{L^p}^{1-p} \left| \frac{1}{p} \int_{\mathbb{T}^2} u_n \cdot \nabla \tilde{q}_n^p \, \mathrm{d}x - \int_{\mathbb{T}^2} Q_n(u_n \cdot \nabla \tilde{q}_n) \, \tilde{q}_n^{p-1} \, \mathrm{d}x \right| \\ &\leq \frac{1}{p} \, \|\nabla u_n\|_{L^\infty} \, \|\tilde{q}_n\|_{L^p} + \|Q_n(u_n \cdot \nabla \tilde{q}_n)\|_{L^\infty} \,. \end{aligned} \tag{80}$$

Now, by the Sobolev embedding theorem, there exists a constant c_1 such that for every $f \in H^2(\mathbb{T}^2)$,

$$\|Q_n f\|_{L^\infty} \leq c_1 \, \|Q_n f\|_{H^{3/2}} \leq \frac{c_1}{\sqrt{n}} \, \|f\|_{H^2} \,, \tag{81}$$

so that

$$\|Q_n(u_n\cdot\nabla\tilde q_n)\|_{L^\infty} \le \frac{c_1}{\sqrt{n}}\,\|u_n\cdot\nabla\tilde q_n\|_{H^2} \le \frac{c_2}{\sqrt{n}}\,\|u_n\|_{H^4}\,\|q_n\|_{H^3}\,. \tag{82}$$

Further, due to Theorem 1, $\|u_n\|_{H^4} \le C(\frac{r+3}{4}, R)$, so that (80) implies

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\tilde q_n(t)\|_{L^p} \le \frac{C_1}{p}\,\|\tilde q_n\|_{L^p} + \frac{C_2}{\sqrt{n}}\,. \tag{83}$$

Integrating this differential inequality in time, we find that for all $t \in [0, T_n]$,

$$\|\tilde q_n(t)\|_{L^p} \le \|\tilde q_n^{\mathrm{in}}\|_{L^p}\,\exp\Big(\frac{C_1\,t}{p}\Big) + \frac{C_2\,p}{C_1\sqrt{n}}\left[\exp\Big(\frac{C_1\,t}{p}\Big) - 1\right]. \tag{84}$$

Letting $p \to \infty$, we obtain

$$\|\tilde q_n(t)\|_{L^\infty} \le \|\tilde q_n^{\mathrm{in}}\|_{L^\infty} + \frac{C_2\,t}{\sqrt{n}} \le \frac{1+r}{2} + \frac{C_2\,\hat T}{\sqrt{n}}\,. \tag{85}$$

Thus, for N large enough and for all $n \ge N$, we have proved that $\|\tilde q_n(t)\|_{L^\infty} < \frac{r+3}{4}$ so long as $\|\tilde q_n(t)\|_{H^m} \le R$.

We can now proceed with a standard estimate on the H^m norm of $\tilde q_n$. First, recall that there are constants c_1 and c_2 such that for all $u, q, \psi \in H^{m+1}(\mathbb{T}^2)$,

$$\langle u\cdot\nabla q, q\rangle_m \le c_1\,\|u\|_{H^{m+1}}\,\|q\|_{H^m}^2\,, \tag{86a}$$

$$\langle u\cdot\nabla q, \psi\rangle_m \le c_2\,\|u\|_{H^{m+1}}\,\|q\|_{H^m}\,\|\psi\|_{H^{m+1}}\,. \tag{86b}$$

Taking the H^m inner product of (78a) with $\tilde q_n$ and using estimates (86a) and (60) yields

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\tilde q_n\|_{H^m}^2 = -2\,\langle u_n\cdot\nabla\tilde q_n, \tilde q_n\rangle_m \le C(r)\,\|\tilde q_n\|_{H^m}^3\,. \tag{87}$$

Integrating the differential inequality (87) with respect to time and recalling that $\|\tilde q_n^{\mathrm{in}}\|_{H^m} \le \|\tilde q^{\mathrm{in}}\|_{H^m}$, we find that there exists a continuous increasing function θ independent of $n \ge N$ with $\theta(0) = \|\tilde q^{\mathrm{in}}\|_{H^m}$ such that

$$\|\tilde q_n(t)\|_{H^m} \le \theta(t) \tag{88}$$

for all $t \in [0, T_n)$. The claim follows immediately. □

Step 3. *Show that $\{q_n\}$ is a relatively compact set in $C([0,T); \text{w-}H^m(\mathbb{T}^2))$ for some $T > 0$.*

Proof. We set $T = \inf_{n\ge N} T_n$. According to the Arzela–Ascoli theorem, $\{q_n\}$ is a relatively compact set in the space $C([0,T); \text{w-}H^m(\mathbb{T}^2))$, where w-$H^m(\mathbb{T}^2)$ denotes H^m endowed with the weak topology, provided the following is true:

1. $\{q_n(t)\}$ is a relatively compact set in w-$H^m(\mathbb{T}^2)$ for every $t \in [0, T)$;

2. $\{q_n\}$ is equicontinuous in $C([0,T);\text{w-}H^m(\mathbb{T}^2))$, i.e. for every $\psi \in H^m$ the sequence $\{\langle \psi, q_n \rangle_m\}$ is equicontinuous in $C([0,T))$.

The first condition is equivalent to $\{q_n(t)\}$ being bounded in H^m for every $t \in [0,T)$, hence is a consequence of (88). To show equicontinuity, we first assume that ψ is smooth. Due to (86b) and (60), we obtain

$$\begin{aligned} \bigl|\langle \psi, q_n(t_2)\rangle_m - \langle \psi, q_n(t_1)\rangle_m\bigr| &= \left| \int_{t_1}^{t_2} \bigl\langle P_n\psi, u_n \cdot \nabla q_n \bigr\rangle_m \,\mathrm{d}t \right| \\ &\le C_1(r) \int_{t_1}^{t_2} \|P_n \psi\|_{H^{m+1}} \, \|q_n\|_{H^m}^2 \,\mathrm{d}t \\ &\le C_2(r,R) \int_{t_1}^{t_2} \|\psi\|_{H^{m+1}} \,\mathrm{d}t \,. \end{aligned} \tag{89}$$

The right side has an upper bound which is proportional to $|t_2 - t_1|$ independent of n, which implies that the set $\{\langle \psi, q_n \rangle_m\}$ is equicontinuous in $C([0,T))$. The class of test functions can be extended to $\psi \in H^m$ by a straightforward density argument. □

Step 4. *Pass to the limit.*

Step 3 asserts the existence of a subsequence, for convenience still denoted $\{q_n\}$, which has a limit q in the topology of $C([0,T);\text{w-}H^m(\mathbb{T}^2))$. Furthermore, weak semicontinuity of the H^m norm and the Rellich–Kondrachov theorem imply that $\tilde{q}(t) \in A^m(\frac{r+7}{8}, 2R)$ for all $t \in [0,T)$.

Thus, $u = K(q)$ is well-defined and we can proceed to show that q and u satisfy the weak vorticity equation

$$\langle \psi, q(t_2) \rangle - \langle \psi, q(t_1) \rangle - \int_{t_1}^{t_2} \bigl\langle \nabla \cdot (u\psi), q \bigr\rangle \,\mathrm{d}t = 0 \tag{90}$$

for every $\psi \in C^\infty(\mathbb{T}^2)$. Indeed, since the embedding $\text{w-}H^m(\mathbb{T}^2) \hookrightarrow L^2(\mathbb{T}^2)$ is continuous,

$$\langle \psi, q_n(t) \rangle \to \langle \psi, q(t) \rangle \tag{91}$$

for every $t \ge 0$ as $n \to \infty$. Moreover, this embedding is compact, so that, using the uniform continuity of K asserted by Theorem 1,

$$\begin{aligned} &\int_{t_1}^{t_2} \Bigl(\bigl\langle \nabla \cdot (u_n \psi), q_n \bigr\rangle - \bigl\langle \nabla \cdot (u\psi), q \bigr\rangle \Bigr) \,\mathrm{d}t \\ &\quad \le \|\nabla \psi\|_{L^\infty} \int_{t_1}^{t_2} \Bigl(\|u_n - u\|_{H^1} \|q_n\|_{L^2} + \|u\|_{L^2} \|q_n - q\|_{L^2} \Bigr) \,\mathrm{d}t \\ &\quad \to 0 \end{aligned} \tag{92}$$

as $n \to \infty$. To prove that q satisfies the vorticity equation (1a) in the classical sense, we must first assert more regularity.

Step 5. *Show that $q \in C([0,T);H^m(\mathbb{T}^2))$ and $u \in C([0,T);H^{m+1}(\mathbb{T}^2))$.*

Proof. From (88) we have that $\|q_n(t)\|_{H^m}$ is bounded by $\theta(t)$, so that the limiting function must also satisfy $\|q(t)\|_{H^m} \leq \theta(t)$. This implies

$$\limsup_{t \searrow 0} \|q(t)\|_{H^m} \leq \theta(0) = \|q^{\text{in}}\|_{H^m} \, . \tag{93}$$

On the other hand, the weak continuity into H^m implies

$$\liminf_{t \searrow 0} \|q(t)\|_{H^m} \geq \|q^{\text{in}}\|_{H^m} \tag{94}$$

and, thus,

$$\lim_{t \searrow 0} \|q(t)\|_{H^m} = \|q^{\text{in}}\|_{H^m} \, . \tag{95}$$

Continuity of the norm and weak continuity imply strong continuity of q as a map into H^m at $t = 0$.

By considering the initial value problem with $q^{\text{in}} \equiv q(t_0)$, the above argument readily shows that q is continuous from the right at any $t_0 \in [0, T)$. But since all our estimates are invariant under time reversal, it must also be continuous from the left. Continuity of u follows from the uniform continuity of K. Furthermore, $u \in C([0,T); H^{m+3}(\mathbb{T}^2))$ if, in addition, $\mu = 0$. □

Step 6. *Show that* $q \in C^k([0,T); H^{m-k})$*,* $h \in C^k([0,T); H^{m+2-k})$*, and* $u \in C^k([0,T); H^{m+1-k})$ *for* $1 \leq k \leq m-2$*.*

Proof. First, consider the case $k = 1$. By assumption, H^{m-1} is a topological algebra; therefore, $q \mapsto u \cdot \nabla q$ is continuous as a map from H^m into H^{m-1}. Since

$$\partial_t q = -u \cdot \nabla q \, , \tag{96}$$

one has $\partial_t q \in C([0,T); H^{m-1})$. Differentiating (2a) in time, we obtain

$$\partial_t h = -L_q^{-1} (h \, \partial_t q) \, . \tag{97}$$

Hence, $h \in C^1([0,T); H^{m+1})$ by Proposition 1. Finally, differentiating (2b) in time, we obtain

$$\partial_t u = \varLambda_h^{-1} \nabla^\perp \partial_t \bigl(h - \mu \, (2 \, h \, \varDelta h + |\nabla h|^2) \bigr) + \sigma \, \varLambda_h^{-1} \bigl(\partial_t h \, \varDelta u + 2 \, \nabla \partial_t h \cdot \nabla u \bigr) \, . \tag{98}$$

Thus, by Theorem 1, $u \in C^1([0,T); H^{m+2})$ if $\mu = 0$ and $u \in C^1([0,T); H^m)$ in the general case. The cases when $k \geq 2$ can be obtained by successive time differentiation of (96), (97), and (98). The necessary estimates remain valid so long as H^{m-k} is a topological algebra. □

Step 7. *Show that the solution is unique.*

Proof. Consider two pairs (u_1, q_1) and (u_2, q_2) of solutions to (1) such that $\tilde{q}_1(t), \tilde{q}_2(t) \in A^m(r, R)$ for $t \in [0, T]$. Setting $q \equiv q_1 - q_2$, we compute

$$\begin{aligned}
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|q\|_{H^{m-1}}^2 &= \langle u_2 \cdot \nabla q, q\rangle_{m-1} + \langle (u_2 - u_1) \cdot \nabla q_1, q\rangle_{m-1} \\
&\leq c_1 \|u_2\|_{H^m} \|q\|_{H^{m-1}}^2 + c_2 \|u_2 - u_1\|_{H^{m-1}} \|q_1\|_{H^m} \|q\|_{H^{m-1}} \\
&\leq c_3 \|q\|_{H^{m-1}}^2 \,, \qquad (99)
\end{aligned}$$

where the first inequality is due to (86) and the fact that H^{m-1} is a topological algebra, and the second inequality is due to the uniform continuity of K asserted by Theorem 1. The resulting differential inequality shows that $q \equiv 0$ on $[0, T]$ provided it is initially so. □

5. Global classical solutions

Lemma 3. *Suppose $\tilde{q} \in H^{m+1}(\mathbb{T}^2)$ with $\|\tilde{q}\|_{L^\infty} < 1$ and let $u = K(q)$. Then there exists a constant C depending on $(1 - \|\tilde{q}^{\mathrm{in}}\|_{L^\infty})^{-1}$ and the parameters such that*

$$\langle u \cdot \nabla q, q\rangle_m \leq C \left(1 + \|\nabla u\|_{L^\infty}\right) \|q\|_{H^m}^2 \,. \qquad (100)$$

Proof. We begin by noting that, for any multi-index α,

$$\left| \int_{\mathbb{T}^2} u \cdot \nabla(\mathrm{D}^\alpha q)\, \mathrm{D}^\alpha q \,\mathrm{d}x \right| = \left| \frac{1}{2} \int_{\mathbb{T}^2} \nabla \cdot u \,(\mathrm{D}^\alpha q)^2 \,\mathrm{d}x \right| \leq \tfrac{1}{2} \|\nabla u\|_{L^\infty} \|q\|_{H^{|\alpha|}}^2 \,. \qquad (101)$$

Thus, using (101), Hölder inequalities, and estimate (60) on the potential vorticity inversion, we estimate

$$\begin{aligned}
\langle u \cdot \nabla q, q\rangle_m &= \sum_{|\alpha| \leq m} \int_{\mathbb{T}^2} u \cdot \nabla(\mathrm{D}^\alpha q)\, \mathrm{D}^\alpha q \,\mathrm{d}x \\
&\quad + \sum_{|\alpha| \leq m} \sum_{\substack{\beta \leq \alpha \\ |\beta| = 1}} \int_{\mathbb{T}^2} \mathrm{D}^\beta u \cdot \nabla \mathrm{D}^{\alpha-\beta} q \, D^\alpha q \,\mathrm{d}x \\
&\quad + \sum_{|\alpha| \leq m} \sum_{\substack{\beta \leq \alpha \\ |\beta| \geq 2}} \int_{\mathbb{T}^2} \mathrm{D}^\beta u \cdot \nabla \mathrm{D}^{\alpha-\beta} q \, \mathrm{D}^\alpha q \,\mathrm{d}x \\
&\leq c_1 \|\nabla u\|_{L^\infty} \|q\|_{H^m}^2 + c_2 \sum_{k=2}^{m} \|u\|_{W^{k,p_{k-1}}} \|q\|_{W^{m-k+1, r_{k-1}}} \|q\|_{H^m} \\
&\leq c_1 \|\nabla u\|_{L^\infty} \|q\|_{H^m}^2 + C_1 \sum_{k=1}^{m-1} \|q\|_{W^{k,p_k}} \|q\|_{W^{m-k,r_k}} \|q\|_{H^m}
\end{aligned} \qquad (102)$$

where

$$p_k = \frac{2m}{k} \qquad \text{and} \qquad r_k = \frac{2m}{m-k} \tag{103}$$

so that

$$\frac{1}{p_k} + \frac{1}{r_k} + \frac{1}{2} = 1\,. \tag{104}$$

Now, applying Gagliardo–Nirenberg inequalities with dimension independent exponents $\mu_k = k/m$ and $\nu_k = (m-k)/m$ to each of the $W^{k,p}$ norms in (102), we obtain

$$\langle u \cdot \nabla q, q \rangle_m \leq c_1 \, \|\nabla u\|_{L^\infty} \, \|q\|_{H^m}^2 + C_2 \sum_{k=1}^{m-1} \|q\|_{H^m}^{1+\mu_k+\nu_k} \, \|q\|_{L^\infty}^{\mu_k+\nu_k} \,. \tag{105}$$

The claim follows immediately.

Theorem 3. *Let $m \geq 3$. Then for every positive initial potential vorticity $q^{\text{in}} \in H^m(\mathbb{T}^2)$ the generalized LSG equations* (1) *have a unique classical solution of class* (76) *for any $T > 0$. Moreover,*

$$q_+(t) = q_+^{\text{in}} \quad \textit{and} \quad q_-(t) = q_-^{\text{in}}\,, \tag{106a}$$

and there exists a constant C depending on the ratio $q_+^{\text{in}}/q_-^{\text{in}}$, and the parameters of the equation such that

$$\|q(t)/Q\|_{H^m} \leq \|q^{\text{in}}/Q\|_{H^m}^{\exp(CQt)} \tag{106b}$$

with $Q \equiv \frac{1}{2}(q_+^{\text{in}} + q_-^{\text{in}})$ for all $t \geq 0$.

Proof. The global pointwise bound (106a) is an immediate consequence of the advection equation (1a) so long as a classical solution exists. Hence, it is enough to prove the global H^m bound (106b) under the assumption that $Q = 1$ and $\|\tilde{q}^{\text{in}}\|_{L^\infty} < 1$ with a dependence of the constant C only on $(1 - \|\tilde{q}\|_{L^\infty})^{-1}$; the general statement follows from the rescaling argument given in Section 2.

To do so, notice that (106a) implies that

$$\|\tilde{q}(t)\|_{L^\infty} = \|\tilde{q}^{\text{in}}\|_{L^\infty}. \tag{107}$$

Now, use a Gagliardo–Nirenberg inequality with exponent $\theta = 2/(p+2)$ to estimate

$$\|\nabla u\|_{L^\infty} \leq c(p) \, \|\nabla u\|_{H^2}^\theta \, \|\nabla u\|_{L^p}^{1-\theta} \leq c(p) \, \|q\|_{H^2}^\theta \, (C_1 \, p)^{1-\theta} \leq C_2 \, p \, \|q\|_{H^m}^{\frac{2}{p+2}} \,, \tag{108}$$

where the constants C_1 and C_2 depend only on $(1 - \|\tilde{q}\|_{L^\infty})^{-1}$ and the parameters. The second inequality in (108) is due to Theorem 1 and estimate (42) of Proposition 3, and the third inequality follows from the advection of potential vorticity and the fact that the Gagliardo–Nirenberg constant satisfies $c(p) \to 1$ as $p \to \infty$.

Without loss of generality, we may assume that $\log\|q\|_{H^m} \geq 2$. Then, setting $p = \log\|q\|_{H^m}$ in (108) and noting that

$$\|q\|_{H^m}^{\frac{2}{2+\log\|q\|_{H^m}}} \leq \mathrm{e}^2 \,, \tag{109}$$

we find that

$$\|\nabla u\|_{L^\infty} \leq C_4 \log\|q\|_{H^m} \,. \tag{110}$$

Now, using Lemma 3 and this estimate, we obtain

$$\frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} \|q\|_{H^m}^2 = -\langle u \cdot \nabla q, q \rangle_m \leq C_5 \left(1 + \log\|q\|_{H^m}\right) \|q\|_m^2 \,. \tag{111}$$

Integration in time yields (106b).

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