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Lagrangian averaging with geodesic mean

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This paper revisits the derivation of the Lagrangian averaged Euler (LAE), or Euler-α equations in the light of an intrinsic definition of the averaged flow map as the geodesic mean on the volume preserving diffeomorphism group. Under the additional assumption that first-order fluctuations are statistically isotropic and transported by the mean flow as a vector field, averaging of the kinetic energy Lagrangian of an ideal fluid yields the LAE Lagrangian. The derivation presented here assumes an Euclidean spatial domain without boundaries.

1. Introduction

The Lagrangian averaged Euler (LAE) equations

$$\partial_t v - u \times (\nabla \times v) + \nabla p = 0 \,, \tag{1.1a}$$

$$v = (1 - \alpha^2 \Delta) u \,, \tag{1.1b}$$

$$\nabla \cdot u = 0 \,, \tag{1.1c}$$

also known as the Euler-α equations, were introduced by Holm, Marsden, and Ratiu [15] based on structural principles, exploring an analogy between the Hamiltonian structure of one-dimensional nonlinear wave equations and that of fluid dynamics. On a domain Ω in three-dimensional Euclidean space, these equations are the Euler–Poincaré equations corresponding to the Lagrangian

$$L_\alpha = \tfrac{1}{2} \int_\Omega |u|^2 + \alpha^2 \, |\nabla u|^2 \, \mathrm{d}x \,. \tag{1.2}$$

Geometrically, the solutions of (1.1) are geodesics on the group of H^s-class volume-preserving diffeomorphisms $\mathrm{SDiff}(\Omega)$ for $s > \frac{5}{2}$ with respect to a right-invariant H^1-metric [22].

The LAE equations subsequently drew attention due to their possible use in modeling turbulence [1, 6, 11, 19], though this subject remains controversial and several authors report unphysical behavior at least in the context of quasi two-dimensional rotating turbulence [10, 16].

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In parallel, there was a quest quest for a sound, if not rigorous first-principles derivation. The general approach, Lagrangian averaging, is pursued by several authors. It involves decomposing the flow into a mean and fluctuating part, making some closure assumption on the evolution of fluctuations, and finally averaging over the fluctuations in the Lagrange functional which appears in the Hamilton principle for fluids. The details regarding the precise setup and set of assumptions differ, but so far no entirely satisfactory derivation has been achieved.

The strategy of Lagrangian averaging was first laid out by Holm [12, 13], who recognized the close connection between Lagrangian averaging and generalized Lagrangian mean (GLM) theories [2]. To provide closure, Holm assumes that fluctuations can be characterized by a vector field which is parallel-transported by the mean flow—an assumption he refers to as a *Taylor hypothesis* in analogy with G.I. Taylor's observation that turbulent fluctuations are correlated in the downstream direction of a flow [25]. Soward and Roberts [20, 23] argue that a different variational principle should be used in this setting, resulting in a different set of averaged equations. Another derivation following Holm's Taylor hypothesis, with different interpretation, is given in [21].

Marsden and Shkoller [17] propose a different Taylor hypothesis: They assume that fluctuations are characterized by composition with a perturbation map which can be written as the flow of a non-autonomous vector field having the perturbation parameter in place of time. At first order in a small amplitude expansion, fluctuations are assumed to be Lie-transported as a vector field. This implies that the first non-trivial contribution comes from second order terms. Initially, Marsden and Shkoller chose a parallel transport closure at second order. As it turns out, however, this assumption is in conflict with the requirement that fluctuations respect incompressibility. Thus, the authors subsequently relax the second order assumption to a weaker orthogonality condition on the parallel transport term [18]; also see [5] for a generalization to compressible flow.

Gilbert and Vanneste [9] recently revisited GLM theory from the geometric point of view, writing it out in intrinsic, coordinate-free language. Their work exposes that such theories are subject to three types of constraints: intrinsic geometric constraints, constraints which depend on the definition of the mean flow map, and closure assumptions. Intrinsic geometric constraints arise directly from the mathematical formulation of the problem and are not subject to choice. The choices for a definition of the mean are also tightly constrained: To be geometrically meaningful, averaging must be viewed as an operation on flow maps. In addition, the mean flow ought to transport a point such that it remains a "center" of the scatter-cloud of the corresponding members of the ensemble as they are transported by each realization of the fluctuating map. Gilbert and Vanneste discuss four notions of mean flow, the most natural of which is the definition of mean flow as the minimizer of geodesic distance on the volume preserving diffeomorphism group with respect to the L^2-metric, as this coincides with the sense in which the flow of the incompressible Euler equations can be interpreted as geodesic flow [3]. I will therefore focus on this notion of average and only comment on the alternatives briefly.

In this framework, there is only one remaining choice that can be made on the basis of physical modeling or empirical studies: the specification of an equation for the evolution of *first order* fluctuations in a small amplitude expansion of the fluctuating map. In particular, the first order Taylor hypothesis of Marsden and Shkoller is such a choice while second and higher order closure conditions are not—they are slaved to the first order condition through the definition the average for maps.

In this paper, I explore the consequences of this geometric view on the derivation of the LAE equations. Specifically, I will show that the LAE equations can be derived by Lagrangian averaging under the following minimal set of assumptions:

(i) The averaged map is the minimizer of geodesic distance,
(ii) first order fluctuations are statistically isotropic, and
(iii) first order fluctuations are transported by the mean flow as a vector field.

Here, assumption (i) fixes the notion of average for maps, assumption (ii) is an assumption on the ensemble statistics, and assumption (iii) is the dynamical closure.

Assumptions (ii) and (iii) coincide with the assumptions of Marsden and Shkoller [17]. The set of all three assumptions can be shown to imply the second order Taylor hypothesis of [18]. Thus, the main message of this paper can be rephrased as follows: Geometric GLM theory provides an intrinsic justification for Marsden and Shkoller's second order hypothesis which, by itself, does not have an such an intrinsic geometric interpretation and has the drawback of being difficult to justify on physical grounds or by computational studies.

The derivation presented here is entirely formal. Averaging, as is common in the context of GLM theories, is considered in the ensemble sense. The precise definition of the ensemble is not important so long as spatial-temporal isotropy of the fluctuations can be assumed. Further, I assume that Ω is a domain without boundaries and that functions and their derivatives decay sufficiently so that integration by parts is freely permitted. In contrast to [9], physical space-time is Euclidean. The geometry of $\mathrm{SDiff}(\Omega)$ is always non-Euclidean, though, and fully treated as such. It remains open whether the same derivation can be written intrinsically when physical space is a general manifold.

The paper is structured as follows. Section 2 introduces Lagrangian averaging. Section 3 reviews the closure assumptions of Marsden and Shkoller in the notation used here. Section 4 introduces the geodesic mean of an ensemble of flow maps. The main result, the derivation of the LAE Lagrangian under the assumptions stated above, is presented in Section 5. The paper closes with a brief discussion.

2. Lagrangian averaging

In this paper, I will follow the setup of Marsden and Shkoller [17, 18] with only minor changes.

Let $u_\varepsilon = u_\varepsilon(x,t)$ denote the velocity field corresponding to a single realization from an ensemble of turbulent ideal incompressible flows. This velocity field generates a volume preserving flow $\eta_\varepsilon = \eta_\varepsilon(x,t)$ via

$$\dot\eta_\varepsilon = u_\varepsilon \circ \eta_\varepsilon \, . \tag{2.1}$$

The symbol "$\circ$" denotes the composition of maps (with respect to the spatial argument). Now suppose that the flow can be decomposed into a averaged flow η and a fluctuating part ξ_ε via

$$\eta_\varepsilon = \xi_\varepsilon \circ \eta \, , \tag{2.2}$$

where $\xi_\varepsilon|_{\varepsilon=0} = \mathrm{id}$, and $\xi_\varepsilon = \xi_\varepsilon(x,t)$ and $\eta = \eta(x,t)$ are again volume preserving time-dependent maps. Suppose further that η is generated by a mean velocity field $u = u(x,t)$ via

$$\dot\eta = u \circ \eta \tag{2.3}$$

with initial condition $\eta|_{t=0} = \mathrm{id}$ so that $\eta_\varepsilon|_{t=0} = \xi_\varepsilon|_{t=0}$. In this setting, we think of ε as the amplitude of the fluctuations which we assume to be small.

So far, this construction is entirely general and we have not specified how the average is defined. As the maps η_ε do not live in a linear space, we cannot average them directly. Rather, we work with a small amplitude expansion of (2.2) and use an abstract averaging operator $\langle \cdot \rangle$ which acts on vector fields. The details of the averaging operator are not important in the context of this note; we will only need minimal assumptions which we specify later.

The strategy is now the following. Take a Lagrangian corresponding to a single realization of an exact Euler flow,

$$L_\varepsilon = \tfrac{1}{2} \int_\Omega |u_\varepsilon|^2 \, \mathrm{d}x \, , \tag{2.4}$$

and expand the velocity field

$$u_\varepsilon = u + \varepsilon \, u' + \tfrac{1}{2} \, \varepsilon^2 \, u'' + O(\varepsilon^3) \tag{2.5}$$

in powers of ε. Here and in the following, the prime denotes a derivative with respect to ε and omission of the ε-subscript always denotes evaluation at $\varepsilon = 0$. Then,

$$\begin{aligned} L_\varepsilon &= \tfrac{1}{2} \int_\Omega |u|^2 + 2\,\varepsilon\, u \cdot u' + \varepsilon^2 \left(|u'|^2 + u \cdot u''\right) \mathrm{d}x + O(\varepsilon^3) \\ &\equiv L_0 + \varepsilon\, L_1 + \tfrac{1}{2}\,\varepsilon^2\, L_2 + O(\varepsilon^3)\,. \end{aligned} \tag{2.6}$$

We now consider an ensemble of Lagrangians L_ε where the mean flow velocity u, by definition, is deterministic and the random fields u', u'', etc., encode the ensemble statistics. Truncating the series and formally applying the averaging operator, we define the *averaged Lagrangian*

$$\begin{aligned} L &= \tfrac{1}{2} \left\langle \int_\Omega |u|^2 + 2\,\varepsilon\, u \cdot u' + \varepsilon^2 \left(|u'|^2 + u \cdot u''\right) \mathrm{d}x \right\rangle \\ &= \tfrac{1}{2} \int_\Omega |u|^2 + 2\,\varepsilon\, u \cdot \langle u' \rangle + \varepsilon^2 \left(\langle |u'|^2 \rangle + u \cdot \langle u'' \rangle\right) \mathrm{d}x\,. \end{aligned} \tag{2.7}$$

The task is now to define closure assumptions on u' and on the respective averages so that the final averaged Lagrangian depends on the mean flow velocity u only, thereby giving rise to new equations for the mean flow via Hamilton's principle.

It is useful to formulate the problem in terms of purely Eulerian quantities. To do so, we consider ξ_ε as a flow with respect to ε, which we think of as an artificial time parameter, and define a corresponding vector field w_ε via

$$\xi_\varepsilon' = w_\varepsilon \circ \xi_\varepsilon\,. \tag{2.8}$$

Due to (2.2), we can also write

$$\eta_\varepsilon' = w_\varepsilon \circ \eta_\varepsilon\,. \tag{2.9}$$

Differentiating (2.9) with respect to t, and (2.1) with respect to ε, we obtain

$$u_\varepsilon' = \dot{w}_\varepsilon + \nabla w_\varepsilon\, u_\varepsilon - \nabla u_\varepsilon\, w_\varepsilon \equiv \dot{w}_\varepsilon + \mathcal{L}_{u_\varepsilon}\, w_\varepsilon\,, \tag{2.10}$$

where ∇u is read as the matrix $(\nabla u)_{ij} = \partial_j u_i$ so that $(\nabla u\, w)_i = \sum_j w_j\, \partial_j u_i$, and $\mathcal{L}_u\, w$ expresses the Lie derivative of the vector field w in the direction of u. Setting (2.10) and its ε-derivative to zero, we obtain the following expressions for the coefficients of the u_ε-expansion in terms of an expansion of the fluctuation vector field w_ε:

$$u' = \dot{w} + \mathcal{L}_u\, w\,, \tag{2.11a}$$

$$u'' = \dot{w}' + \mathcal{L}_u\, w' + \mathcal{L}_{u'}\, w\,. \tag{2.11b}$$

3. The Marsden–Shkoller–Taylor hypotheses

In the following, we give a brief account of Marsden and Shkoller's [18] derivation of the isotropic LAE equations. They express their closure in terms of the quantities ξ' and ξ'' which are Lagrangian rates of change with respect to amplitude parameter ε. To translate to the expansion coefficients of the Eulerian fluctuation vector field w_ε, we evaluate various mixed derivatives of (2.8) at $\varepsilon = 0$ and note that $\dot{\xi} = 0$:

$$\xi' = w\,, \tag{3.1a}$$

$$\dot{\xi}' = \dot{w}\,, \tag{3.1b}$$

$$\xi'' = w' + \nabla w\, w\,, \tag{3.1c}$$

$$\dot{\xi}'' = \dot{w}' + \nabla \dot{w}\, w + \nabla w\, \dot{w}\,. \tag{3.1d}$$

Then, the expansion coefficients (2.11) take the form

$$u' = \dot{\xi}' + \nabla\xi'\, u - \nabla u\, \xi' \,, \tag{3.2a}$$

$$u'' = \dot{\xi}'' + \nabla\xi''\, u - 2\,\nabla u'\, \xi' - \nabla\nabla u : \xi' \otimes \xi' - \nabla u\, \xi'' \,. \tag{3.2b}$$

Marsden and Shkoller now make the closure assumptions or "generalized Taylor hypotheses"

$$\dot{\xi}' + \nabla\xi'\, u - \nabla u\, \xi' = 0 \,, \tag{3.3a}$$

$$\langle \dot{\xi}'' + \nabla\xi''\, u \rangle \perp u \,. \tag{3.3b}$$

Assumption (3.3*a*) states that the first order fluctuations $w = \xi'$ are Lie-advected as a vector field. It implies that $u' = 0$ in (3.2). Then, inserting (3.2) and (3.3) into (2.7), we obtain

$$\begin{aligned} L &= \tfrac{1}{2} \int_\Omega |u|^2 - \varepsilon^2\, u \cdot \left(\nabla\nabla u : \langle \xi' \otimes \xi' \rangle + \nabla u\, \langle \xi'' \rangle \right) \mathrm{d}x \\ &= \tfrac{1}{2} \int_\Omega |u|^2 + \varepsilon^2\, (\nabla u)^T\, \nabla u : F + \tfrac{1}{2}\, \varepsilon^2\, |u|^2\, \langle \nabla \cdot \xi'' \rangle \, \mathrm{d}x \,, \end{aligned} \tag{3.4}$$

where F is the symmetric 2-tensor

$$F = \langle \xi' \otimes \xi' \rangle = \langle w \otimes w \rangle \,. \tag{3.5}$$

To eliminate ξ'', take the divergence of (3.1*c*) and note that w' is divergence-free by construction, so that

$$\nabla \cdot \xi'' = \nabla w^T : \nabla w \,. \tag{3.6}$$

Integrating by parts again, we obtain

$$L = \tfrac{1}{2} \int_\Omega |u|^2 + \varepsilon^2 \left((\nabla u)^T\, \nabla u + \tfrac{1}{2}\, \nabla\nabla |u|^2 \right) : F \, \mathrm{d}x \,. \tag{3.7}$$

Finally, assuming that the first-order fluctuations are statistically isotropic so that, normalizing to $F = I$, we obtain

$$L = \tfrac{1}{2} \int_\Omega |u|^2 + \varepsilon^2\, |\nabla u| \, \mathrm{d}x \,. \tag{3.8}$$

Thus, we have obtained the LAE Lagrangian (1.2).

4. Geodesic mean

Gilbert and Vanneste [9] make the point that the mean map η should in some sense be close to the ensemble average of the maps η_ε. Moreover, the notion of closeness should be consistent with the geometry of the Euler equations themselves. This singles out the Riemannian center of mass with respect to the right-invariant L^2-metric on the volume-preserving diffeomorphism group $\mathrm{SDiff}(\Omega)$. In this setting, the geodesic distance between two maps $\phi, \psi \in \mathrm{SDiff}(\Omega)$ is given by

$$d^2(\phi, \psi) = \inf_{\substack{\gamma_s :\, [0,1] \to \mathrm{SDiff}(\Omega) \\ \gamma_0 = \phi, \gamma_1 = \psi}} \int_0^1 \int_\Omega |\dot{\gamma}_s|^2 \, \mathrm{d}x \, \mathrm{d}s \,, \tag{4.1}$$

see, e.g., [4]. Then the associated Riemannian center of mass is the map

$$\eta = \operatorname*{arg\,min}_{\phi \in \mathrm{SDiff}(\Omega)} \langle d^2(\phi, \eta_\varepsilon) \rangle \,. \tag{4.2}$$

Gilbert and Vanneste show that single realizations η_ε are reached from η by integrating the Euler equations in fictitious time ε,

$$w_\varepsilon' + \nabla w_\varepsilon\, w_\varepsilon + \nabla \phi_\varepsilon = 0 \,, \tag{4.3a}$$

$$\nabla \cdot w_\varepsilon = 0 \,, \tag{4.3b}$$

together with a constraint on the initial condition

$$\langle w \rangle = 0 \tag{4.3c}$$

where, as before, we write $w \equiv w_\varepsilon|_{\varepsilon=0}$. Once this notion of average is imposed, we retain the freedom to choose $w(\,\cdot\,,t)$ subject to the constraint (4.3*c*). In the next section, we will do so by imposing an evolution equation for w as our closure condition.

5. Lagrangian averaging as a geodesic GLM closure

In the following, I will re-derive the LAE Lagrangian (3.8) keeping the first order Shkoller–Marsden–Taylor hypothesis (3.3*a*) and the assumption of isotropy, but replacing the *ad hoc* second order closure (3.3*b*) by the fluctuation constraint (4.3) implied by the concept of geodesic mean.

Written in terms of the first order fluctuation vector field, hypothesis (3.3*a*) reads

$$\dot w + \mathcal{L}_u \, w = 0 \,. \tag{5.1}$$

When w satisfies $\langle w \rangle = 0$ initially, it will also satisfy this constraint for all later times. As before, using (2.11), the first and second order fluctuations of the velocity field reduce to

$$u' = 0 \,, \tag{5.2a}$$

$$u'' = \dot w' + \mathcal{L}_u \, w' \,. \tag{5.2b}$$

Once (5.1) is imposed, the geodesic mean condition (4.3*a*) determines the evolution of fluctuations to any order in ε. Evaluating (4.3*a*) at $\varepsilon = 0$, inserting this expression into (5.2*b*), and eliminating time derivatives of w via (5.1), we obtain

$$u'' = \nabla(\mathcal{L}_u \, w) \, w + \nabla w \; \mathcal{L}_u \, w - \nabla \dot\phi - \mathcal{L}_u(\nabla w \, w + \nabla \phi) \,. \tag{5.3}$$

This means that the first non-zero contribution to the perturbation series for L_ε reads

$$\begin{aligned} L_2 &= \int_\Omega u \cdot u'' \, \mathrm{d}x \\ &= \int_\Omega u \cdot \big(\nabla(\mathcal{L}_u \, w) \, w + \nabla w \; \mathcal{L}_u \, w - \mathcal{L}_u(\nabla w \, w + \nabla\phi) \big) \, \mathrm{d}x \\ &= \int_\Omega \nabla u \, w \cdot \nabla u \, w - \nabla u \, w \cdot \nabla w \, u + u \cdot \nabla w \, \nabla w \, u - u \cdot \nabla w \, \nabla u \, w \, \mathrm{d}x \\ &\quad + \int_\Omega \nabla u \, u \cdot \nabla w \, w + u \cdot \nabla u \, \nabla w \, w + \nabla u \, u \cdot \nabla \phi + u \cdot \nabla u \, \nabla \phi \, \mathrm{d}x \end{aligned} \tag{5.4}$$

where, in the second equality, we have used that u is divergence free, thus L^2-orthogonal to gradients, and in the third equality we have integrated by parts in the first, second, fifth, and seventh term. The potential ϕ is determined by the "pressure equation" for (4.3),

$$\nabla w^T : \nabla w + \Delta \phi = 0 \,. \tag{5.5}$$

Further, by integrating each term by parts, we can show that

$$\int_\Omega \nabla u \, u \cdot \nabla w \, w - \nabla u \, w \cdot \nabla w \, u + u \cdot \nabla w \, \nabla w \, u - u \cdot \nabla w \, \nabla u \, w \, \mathrm{d}x = 0 \,. \tag{5.6}$$

Removing these terms from the right hand side of (5.4) and further integrating by parts, we find

$$\begin{aligned} L_2 &= \int_\Omega \nabla u \, w \cdot \nabla u \, w - \tfrac12 \, w \otimes w : \nabla\nabla |u|^2 - \phi \, \nabla u^T : \nabla u - \tfrac12 \, \Delta \phi \, |u|^2 \, \mathrm{d}x \\ &= \int_\Omega w \otimes w : (\nabla u_i \otimes \nabla u_i + \nabla\nabla\Delta^{-1}(\nabla u^T : \nabla u)) \, \mathrm{d}x \end{aligned} \tag{5.7}$$

where we have used (5.5) to eliminate ϕ and twice integrated by parts, so that the second and fourth term on the right cancel.

Finally, taking the average under the assumption of isotropy, i.e.,

$$\langle w \otimes w \rangle = I \tag{5.8}$$

and invoking incompressibility one final time, we find that

$$\langle L_2 \rangle = \int_\Omega |\nabla u|^2 \, \mathrm{d}x \tag{5.9}$$

so that the averaged Lagrangian to second order in ε reads

$$L = \tfrac{1}{2} \int_\Omega |u|^2 + \varepsilon^2 \, |\nabla u|^2 \, \mathrm{d}x \, . \tag{5.10}$$

We have obtained, once again, the LAE Lagrangian (1.2).

6. Discussion

I have given a formal derivation of the LAE equations via "Lagrangian averaging" under three natural assumptions: the mean map is a minimizer of geodesic distance, statistical isotropy, and first order fluctuations are transported as a vector field.

At the technical level, the derivation and resulting equations are the same as given by Marsden and Shkoller: It is easy to check that their "second order Taylor hypothesis" (3.3*b*) is implied by the set of assumptions proposed here; the proof amounts to replaying the argument why none but the first term in (5.7) contributes to the averaged Lagrangian. Conceptionally, however, the new derivation is advantageous as it replaces a hypothesis that used to require independent physical or empirical justification by a condition which arises from the rigidity of the geometric framework. It is important to note that (3.3*b*) is only implied under the assumption of isotropy of fluctuations (5.8). When the covariance tensor F is considered as a dynamic quantity, our averaged Lagrangian (5.7) is different from [18, equation 22] and leads to a different evolution equation for the mean flow.

In this light, it is worth looking at the three alternative definitions of the mean flow suggested by Gilbert and Vanneste [9]. The—in their terminology—"extended GLM"-average relaxes the incompressibility constraint on the fluctuation map. Consequently, (4.3) must be replaced by a pressureless Euler equation. As the fictitious pressure ϕ, under the assumption of isotropy, does not contribute to the resulting averaged Lagrangian, the resulting equation for the mean flow will not be affected by this change. The "optimal transport"-average also drops the incompressibility constraint for the fluctuation map, but keeps averages of the first order fluctuation vector field divergence free. As such, it changes the implied assumption on the fluctuation statistics, but also leads to the same equation for the mean flow. Last, "glm"-averaging, due to Soward and Roberts [24], has a trivial second order fluctuation field by construction. In the setting here, this leads to trivial first and second order averaged Lagrangians. In addition, the "glm"-average is not based on minimizing distance and breaks the conceptual symmetry between the t-flow and the ε-flow. As a result, I conclude that "glm" is not an appropriate setting for Lagrangian averaging while the other three definitions lead to the same result with slightly different interpretations of the assumed fluctuation statistics. Geodesic averaging is arguably the most natural choice as it considers fluctuations and mean flow both incompressible. Note, however, that the models resulting from "extended GLM", "optimal transport", and "geodesic mean" will differ when the isotropy assumption is dropped.

The derivation given here does not address the question whether the LAE equations are a good model for turbulence, but it gives a clearer picture regarding the minimal set of underlying assumptions. In particular, the first order Taylor hypothesis may be open to computational verification using highly resolved reference simulations of isotropic turbulence. The derivation also gives a recipe for deriving averaged equations for systems beyond ideal fluid flow so long as they possess an underlying variational principle. In this context, the framework of geometric GLM theory provides additional important constraints on the choice of admissible closures. It is an independent question whether the geometric framework itself is necessary or helpful on physical grounds. Comparisons such as performed by Geurts [8], who sees the α-model as one instance in a more general class of regularization closures, may shed more light on this question.

On the theoretical side, it is open whether the transport of fluctuation assumption is compatible with the ergodicity of turbulence hypothesis [7], or whether isotropy is maintained for time averages. Further, the simplification from (5.4) to (5.7) is surprising and it remains to find a geometric explanation. Finally, recent work of Holm [14] also uses a vector field interpretation for the Lagrangian fluctuations, now in a stochastic integral context. It is an obvious question whether the present view of geometric averaging would also apply in the stochastic context.

Ethics. This work does not involve any living subjects.

Data Accessibility. This work does not involve any data.

Competing Interests. I have no competing interests.

Authors' Contributions. I have written this paper as the sole author.

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