Partial Differential Equations

Homework 5

due October 30, 2002

1. Prove a maximum principle for the following semilinear PDE, called Burger's equation,

$$u_t + u\,u_x = u_{xx}\,,$$
$$u(x,0) = g(x)$$

where $u = u(x,t)$ and $(x,t) \in \mathbb{R} \times [0,\infty)$.

2. Find the Fourier transforms for the following functions on $\mathbb{R}$:

   (a) $f(x) = e^{-t|x|}$,

   (b) $f(x) = \begin{cases} 1 & \text{for } |x| \le 1 \\ 0 & \text{for } |x| > 1 \end{cases}.$

   Use the result from (a) to solve the 1-dimensional differential equation

$$u_{xx} - u = \delta\,,$$

   where δ is the Dirac measure centered at $x = 0$.

3. Let $f, g \in L^1(\mathbb{R}^n)$, i.e.

$$\|f\|_{L^1} \equiv \int_{\mathbb{R}^n} |f(x)|\,dx < \infty\,;$$

   similarly for g. Show

   (a) $\displaystyle\lim_{y\to 0} \int_{\mathbb{R}^n} |f(x) - f(x-y)|\,dx = 0.$

   (b) $\|f * g\|_{L^1} \le \|f\|_{L^1}\,\|g\|_{L^1}$

   (c) Suppose that, moreover, $g \in L^\infty(\mathbb{R}^n)$. Conclude that $f * g \in C(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)$.

4. Use the Fourier transform to re-derive the fundamental solution of the heat equation.