Numerical Methods I

Midterm Exam

October 17, 2003

1. In the following transcript of an Octave session, all output has been deleted.

```
octave:1> a=1e308; b=1.05e308; c=-1.5e308;
octave:2> a+(b+c)
ans = ..................
octave:3> (a+b)+c
ans = ..................
octave:4> a/((1+a)-a)
ans = ..................
octave:5> x=1e-15;
octave:6> 1-cos(x)
ans = ..................
octave:7> sin(x)^2/(1+cos(x))
ans = ..................
```

   Identify which of the following answers belongs to each of the dotted lines.

   (a) `ans = 0`

   (b) `ans = Inf`

   (c) `ans = 5.0000e-31`

   (d) `ans = 5.5000e+307`

   (e) `warning:  division by zero`
       `ans = Inf`

   (10)

2. Suppose that a function has a zero in the interval $[0, 1]$. Show that the bisection method is guaranteed to approximate the zero within a specified tolerance ε after

$$k \geq \frac{\ln 1/\varepsilon}{\ln 2} - 1$$

   iterations. (10)

3. Consider the matrix
$$A = \begin{pmatrix} 1 & \varepsilon \\ \varepsilon & 1 \end{pmatrix} .$$

   (a) Compute the condition number of A in a matrix norm of your choice. What happens if ε is close to 1?

   (b) Show that the Jacobi method for solving $A\boldsymbol{x} = \boldsymbol{b}$ converges when $|\varepsilon| < 1$.

   (c) Show that the Gauss–Seidel method for solving $A\boldsymbol{x} = \boldsymbol{b}$ converges when $|\varepsilon| < 1$. Which method converges faster?

   Hint: Recall that the Gauss–Seidel method is based on the splitting $A = P + (A - P)$ where P contains the right upper triangular entries of A. So
$$P = \begin{pmatrix} 1 & \varepsilon \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad P^{-1} = \begin{pmatrix} 1 & -\varepsilon \\ 0 & 1 \end{pmatrix} .$$

(10+10+10)

4. Compute the LU decomposition of the matrix
$$A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} .$$

(10)

5. Consider the sequence
$$I_0 = 1 - e^{-1} ,$$
$$I_n = 1 - n\, I_{n-1} .$$

   (a) What is the absolute condition number of computing I_n in terms of I_{n-1}?

   What is the absolute condition number of recursively computing I_n in terms of I_0?

   (b) Is this recursion a good method to compute I_n? Explain!

   Hint: $I_n \to 0$ as $n \to \infty$; see below.

   (c) **Extra credit:** Show that $I_n \to 0$ as $n \to \infty$.

   Hint: Show that
$$I_n = \int_0^1 (1 - x)^n \, e^{-x} \, dx .$$

(10+10+*10*)

6. The following root finding method is a modification of the bisection method. It is called *regula falsi*.
$$x_{k+1} = x_0 - \frac{x_k - x_0}{f(x_k) - f(x_0)} \, f(x_0) . \qquad (*)$$

(a) Show that the *regula falsi* is consistent.

(Recall that a method is consistent if every fixed point ξ of this iteration solves the equation $f(\xi) = 0$.)

(b) Give an argument using Taylor expansion that the *regula falsi* is convergent with order 1.

(c) **Extra credit:** The *regula falsi* is applied in the following way. Choose two starting values x_0 and x_1 so that $f(x_0) \cdot f(x_1) < 0$. Compute the sequence of x_k via (*). If $f(x_0) \cdot f(x_{k+1}) > 0$, re-initialize by setting $x_0 := x_k$.

Show that if f is continuous, the sequence x_k will always converge to a root of f.

Hint: Notice that x_{k+1} is the zero of the line joining the points $(x_0, f(x_0))$ and $(x_k, f(x_k))$. Then note that the sequence of intervals that bracket the root has monotonic bounds.

(10+10+*10*)