Numerical Methods I

Problem Set 10

due in class, December 3, 2003

1. **Project:** Implement each of the following solvers as an Octave function. Each solver should take arguments of the form

   ```
   ode_rk4 ('f', [t0,t1], y0, N)
   ```

   where `f` is the function $f(t,y)$ on the right hand side of the differential equation, `t0` is the initial time, `t1` is the final time, `y0` the initial condition, and `N` the number of steps.

   (a) `ode_midpoint`, the implicit midpoint method from Lab 8, where

   $$\boldsymbol{y}_{n+1} = \boldsymbol{y}_n + h\,\boldsymbol{f}\Big(\frac{t_n + t_{n+1}}{2}, \frac{\boldsymbol{y}_n + \boldsymbol{y}_{n+1}}{2}\Big) .$$

   Note: Use simple iteration to solve for $\boldsymbol{y}_{n+1}$.

   (b) `ode_rk4`, one of the fourth order explicit Runge–Kutta methods given by

   $$\begin{aligned} \boldsymbol{k}_1 &= \boldsymbol{f}(t_n, \boldsymbol{y}_n) , \\ \boldsymbol{k}_2 &= \boldsymbol{f}(t_n + \tfrac{1}{2}h, \boldsymbol{y}_n + \tfrac{1}{2}h\,\boldsymbol{k}_1) , \\ \boldsymbol{k}_3 &= \boldsymbol{f}(t_n + \tfrac{1}{2}h, \boldsymbol{y}_n + \tfrac{1}{2}h\,\boldsymbol{k}_2) , \\ \boldsymbol{k}_4 &= \boldsymbol{f}(t_{n+1}, \boldsymbol{y}_n + h\,\boldsymbol{k}_3) , \\ \boldsymbol{y}_{n+1} &= y_n + \tfrac{1}{6}h\,(\boldsymbol{k}_1 + 2\,\boldsymbol{k}_2 + 2\,\boldsymbol{k}_3 + \boldsymbol{k}_4) . \end{aligned}$$

   (c) `ode_bdf4`, the fourth order backward differentiation formula

   $$25\,\boldsymbol{y}_{n+4} - 48\,\boldsymbol{y}_{n+3} + 36\,\boldsymbol{y}_{n+2} - 16\,\boldsymbol{y}_{n+1} + 3\,\boldsymbol{y}_n = 12h\,\boldsymbol{f}(t_{n+4}, \boldsymbol{y}_{n+4}) .$$

   Note: Use `ode_rk4` to generate the required three extra initial values.

2. **Project:** Verify, as on the previous homework, the order of each of the integrators on a model problem.

3. Analyze the linear stability of critical points of the Volterra–Lotka model

   $$\begin{aligned} \dot{x} &= x\,(y-1) \equiv f(x,y) , \\ \dot{y} &= y\,(1-x) \equiv g(x,y) , \end{aligned}$$

   as follows.

(a) Find the critical points, i.e. the zeros of the right hand side of the equation. (There are two critical points in this case.)

(b) Compute the Jacobi Matrix of the right hand side, i.e., the matrix of partial derivatives

$$DF = \begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{pmatrix} .$$

(c) Find the eigenvalues of DF evaluated at each of the critical points. If all the eigenvalues have negative real part, then the critical point is *linearly stable*, if at least one eigenvalue has positive real part, the critical point is *linearly unstable*, if all the eigenvalues have zero real part, the critical point is *linearly neutrally stable*.

(You should fine one linearly neutrally stable and one linearly unstable critical point.)

4. **Project:** It can be shown that the critical point at $(1, 1)$ of the Volterra–Lotka model is actually stable in the sense that if a solution starts near this point, it will always remain in a small neighborhood of this point.

   Test the three numerical methods you have implemented to see if they retain this property.

5. **Project:** (See QS, Section 7.9) Solve the Van der Pol equation

$$\begin{aligned} \dot{x} &= y \, , \\ \dot{y} &= \mu \, (1 - x^2) \, y - x \, , \end{aligned}$$

   with initial data $x(0) = 2$ and $y(0) = 0$. Which of the methods works best when μ is large?