

Numerical Methods I

Problem Set 5

due in class, October 15, 2003

1. Consider the following class of iterative methods for solving $A\mathbf{x} = \mathbf{b}$, called *stationary Richardson methods*:

$$\mathbf{x}_{k+1} = \mathbf{x}_k + \alpha P^{-1}\mathbf{r}_k$$

where $\mathbf{r}_k = \mathbf{b} - A\mathbf{x}_k$ is the residual in the k th step. Assume that all eigenvalues of $P^{-1}A$ are real and positive, and let λ_{\min} and λ_{\max} denote the minimum and maximum eigenvalue, respectively.

- (a) Show that the method converges if and only if $\alpha < 2/\lambda_{\max}$.
- (b) Show that the method converges fastest if

$$\alpha = \frac{2}{\lambda_{\min} + \lambda_{\max}}.$$

2. Given any iterative method for solving $A\mathbf{x} = \mathbf{b}$ of the form

$$\mathbf{x}_{k+1} = \mathbf{x}_k + B\mathbf{r}_k$$

where, again, $\mathbf{r}_k = \mathbf{b} - A\mathbf{x}_k$. Show that the method is consistent for any nonsingular B .

(Recall: a method is consistent if any fixed point of the method is a solution to $A\mathbf{x} = \mathbf{b}$.)

3. A matrix A is called *strictly diagonally dominant by rows* if

$$|a_{ii}| > \sum_{\substack{j=1 \\ j \neq i}}^n |a_{ij}|$$

for every $i = 1, \dots, n$.

Show that if A has this property, then the Jacobi method converges.

4. **Project:** Write an Octave code to solve $A\mathbf{x} = \mathbf{b}$ by each of the the following methods:

- (a) Jacobi method
- (b) Gradient method

Plot the 2-norm of the residual on a logarithmic scale vs. the number of iterations for the following $n \times n$ test problem (as in Lab 4) for $n = 10$:

$$A = \begin{pmatrix} 2 & -1 & 0 & & \cdots & & 0 & -\frac{1}{2} \\ -1 & 2 & \ddots & \ddots & & & & 0 \\ 0 & \ddots & \ddots & & & & & \vdots \\ \vdots & & \ddots & & & & \ddots & \\ 0 & & & & \ddots & \ddots & \ddots & 0 \\ -\frac{1}{2} & 0 & & \cdots & & 0 & -1 & 2 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}.$$

Which method works best?