Numerical Methods I

Problem Set 8

due in class, November 19, 2003

1. (From SM.) A quadrature formula on the interval $[-1, 1]$ uses the quadrature points $x_0 = -\alpha$ and $x_1 = \alpha$, where $0 < \alpha \leq 1$:

$$\int_{-1}^{1} f(x)\,\mathrm{d}x \approx w_0\, f(-\alpha) + w_1\, f(\alpha)\,.$$

   (a) The formula is required to be exact whenever f is a polynomial of degree 1. Show that $w_0 = w_1 = 1$, independent of the value of α.

   (b) Show that there is one particular value of α for which the formula is exact for all polynomials of degree 2. Find this α, and show that, for this value, the formula is also exact for all polynomials of degree 3.

2. Consider the composite trapezoidal rule for evaluating the integral

$$\int_{0}^{1} x^{1/3}\,\mathrm{d}x\,.$$

   (a) Show, by explicit evaluation, that the local error on the interval $[0, h]$ is proportional to $h^{4/3}$.

   (b) Show that the global error is also proportional to $h^{4/3}$.
   Hint: Use part (a) on the first partition and one of the standard error estimates on all other partitions.

3. **Project:** Use the composite trapezoidal rule with N partitions to approximate the integral of $f(x) = \sinh x$ and $g(x) = \cosh x$ on the interval $[-1, 1]$. As in Lab 7, generate a doubly logarithmic error plot. Which of the functions is integrated more accurately?

4. Explain the behavior seen in the previous question using the Euler–Maclaurin summation formula.

5. **Project:** Use Romberg integration to compute the integral of

$$\begin{aligned} f(x) &= \mathrm{e}^x\,, \\ g(x) &= \sin 2\pi x\,, \\ h(x) &= x^{1/3} \end{aligned}$$

on the interval $[0,1]$. Generate a doubly logarithmic error plot and compare with the results from Lab 7.

6. (From SM.) Show that the weights in the Gauss quadrature formula can also be computed via

$$W_k = \int_a^b w(x)\, L_k(x)\, \mathrm{d}x\,.$$

Recall: Gauss quadrature is based on the expression

$$\int_a^b w(x)\, f(x)\, \mathrm{d}x \approx \sum_{k=0}^n W_k\, f(x_k) + \sum_{k=0}^n V_k\, f'(x_k)\,,$$

where

$$\begin{aligned} W_k &= \int_a^b w(x)\, H_k(x)\, \mathrm{d}x\,, \\ V_k &= \int_a^b w(x)\, K_k(x)\, \mathrm{d}x\,, \end{aligned}$$

and where H_k and K_k are the Hermite interpolation basis polynomials, which can be written in terms of the Lagrange interpolation basis polynomials L_k as

$$\begin{aligned} H_k(x) &= L_k^2(x)\,\bigl(1 - 2L_k'(x_k)\,(x - x_k)\bigr)\,, \\ K_k(x) &= L_k^2(x)\;(x - x_k)\,. \end{aligned}$$

The Gauss quadrature points $x_0, \dots, x_n$ are chosen such that $V_k = 0$ for all $k = 1, \dots, n$.