Numerical Methods I

Problem Set 9

due in class, November 26, 2003

1. The Chebychev polynomial of degree n is given by
$$T_n(x) = \cos(n \arccos x)\,.$$
   (a) Show that the extrema of T_n are located at
$$x_k = \cos\frac{k\pi}{n}$$
   for $k = 0, \ldots, n$, and that the zeros of T_n are located at
$$x_k = \cos\frac{(1+2k)\pi}{2n}$$
   for $k = 0, \ldots, n-1$.
   (b) Show that T_n and T_m are orthogonal with respect to the inner product
$$\langle f, g\rangle = \int_{-1}^{1} \frac{f(x)\,g(x)}{\sqrt{1-x^2}}\,\mathrm{d}x$$
   if and only if $m \neq n$.
   Hint: Use trigonometric substitution, then note that
$$2\,\cos\alpha\,\cos\beta = \cos(\alpha+\beta) + \cos(\alpha-\beta)\,.$$
2. Solve the logistic differential equation
$$\dot{y} = y - y^2\,,$$
$$y(0) = y_0\,.$$
3. Compute the matrix exponential $\exp(A)$ for
   (a) $A = \begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix}$,
   (b) $A = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$.
4. Give an example that $\exp(A+B) \neq \exp(A)\,\exp(B)$.
5. **Project:** Solve the logistic equation from Question 2 on the interval $[0,1]$ by using the explicit Taylor methods of order 1 and 2. Generate a doubly logarithmic error plot to demonstrate the order of the methods.