Solutions to Homework 6

1. Number of operations for standard LU: $\approx \frac{2}{3} n^3$ (SM, p. 57)

   for Cholesky : $\approx \frac{1}{3} n^3$ (SM, p. 93)

For each step of CG we have to compute

r_k : $n^2 + O(n)$ operations

$A p_{k-1}$: n^2 operations, but can be reused from previous step

p_k : $O(n)$ operations once $A p_{k-1}$ is known

$A p_k$: n^2 operations

α_k : $O(n)$ operations once $A p_k$ is known

x_{k+1} : $O(n)$ operations

Thus, an efficient implementation will have $2n^2 + O(n)$ operations per step, so n steps take $2n^3 + O(n^2)$ ops; 6 times longer than Cholesky.

<u>Note</u>: In all this, it is common practice to count 1 addition and one multiplication as a single operation.

3. a) $L_0(x) = \dfrac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)}$

$L_1(x) = \dfrac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)}$

$L_2(x) = \dfrac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)}$

$$p(x) = L_0(x)\underbrace{f(x_0)}_{=:y_0} + L_1(x)\underbrace{f(x_1)}_{=:y_1} + L_2(x)\underbrace{f(x_2)}_{=:x_2}$$

(Further simplifications do not gain much.)

b) $$f'(x) \approx p'(x) = \frac{(x-x_1)+(x-x_2)}{(x_0-x_1)(x_0-x_2)} y_0 + \frac{(x-x_0)+(x-x_2)}{(x_1-x_0)(x_1-x_2)} y_1 + \frac{(x-x_0)+(x-x_1)}{(x_2-x_0)(x_2-x_1)} y_2$$

$$\Rightarrow f'(x_1) \approx \frac{(x_1-x_2)}{(x_0-x_1)(x_0-x_2)} y_0 + \left(\frac{1}{x_1-x_2} + \frac{1}{x_1-x_0}\right) y_1 + \frac{x_1-x_0}{(x_2-x_0)(x_2-x_1)} y_2$$

$$f''(x) \approx p''(x) = 2\left[\frac{y_0}{(x_0-x_1)(x_0-x_2)} + \frac{y_1}{(x_1-x_0)(x_1-x_2)} + \frac{y_2}{(x_2-x_0)(x_2-x_1)}\right]$$

c) Equidistant nodes: $x_1 - x_0 = h = x_2 - x_1$; $x_2 - x_0 = 2h$

$$\Rightarrow f'(x_1) \approx \frac{-h}{(-h)(-2h)} y_0 + \left(\frac{1}{-h} + \frac{1}{h}\right) y_1 + \frac{h}{2h \cdot h} y_2$$

$$= \frac{y_2 - y_0}{2h}$$

$$f''(x_1) \approx 2\left[\frac{y_0}{(-h)(-2h)} + \frac{y_1}{h\cdot(-h)} + \frac{y_2}{2h \cdot h}\right] = \frac{y_0 - 2y_1 + y_2}{h^2}$$

4. It is easy to see that $p(x)$ is a polynomial of degree at most $n+1$. Therefore, if $p(x)$ interpolates all $n+2$ points, it is the Lagrange polynomial (the <u>unique</u> polynomial of degree at most $n+1$ with this property).

Check: • $p(x_0) = \dfrac{(x_0 - x_0)\, r(x_0) - (x_0 - x_{n+1})\, q(x_0)}{x_{n+1} - x_0}$

$= q(x_0) = y_0$ ✓ (as q interpolates at x_0)

• $p(x_{n+1}) = \dfrac{(x_{n+1} - x_0)\, r(x_{n+1}) - (x_{n+1} - x_{n+1})\, q(x_{n+1})}{x_{n+1} - x_0}$

$= r(x_{n+1}) = y_{n+1}$ ✓ (as r interpolates at x_{n+1}

• For $j = 1, \dots, n$: $r(x_j) = q(x_j) = y_j$, so that

$p(x_j) = \dfrac{(x_j - x_0)\, r(x_j) - (x_j - x_{n+1})\, q(x_j)}{x_{n+1} - x_0}$

$= \dfrac{(x_j - x_0)\, y_j - (x_j - x_{n+1})\, y_j}{x_{n+1} - x_0}$

$= y_j$ ✓