Partial Differential Equations

Homework 3

due September 28, 2004

1. Prove that the Taylor series with integral remainder for a function $u \in C^N(\mathbb{R}^n)$ is

$$u(x) = \sum_{|\alpha| \leq N-1} \frac{D^\alpha u(x_0)}{\alpha!} (x - x_0)^\alpha + R_N,$$

   where

$$R_N = N \sum_{|\alpha|=N} \frac{(x-x_0)^\alpha}{\alpha!} \int_0^1 (1-t)^{N-1} D^\alpha u(x_0 + t(x-x_0)) \, dt.$$

   Hint: Apply the one-dimensional Taylor formula with integral remainder to the function $f(s) = u(x_0 + s(x - x_0))$.

2. In class we have used Liouville's theorem to show that any bounded solution of the Poisson equation

$$-\Delta u = f$$

   for $f \in C^2_{\mathrm{c}}(\mathbb{R}^n)$, $n \geq 3$, is given by the solution formula

$$u(x) = \int_{\mathbb{R}^n} \Phi(x-y) \, f(y) \, dy$$

   up to an arbitrary additive constant. (Evans, p. 30, Theorem 8.)

   This statement does not hold in dimension $n = 2$ since solutions are generically unbounded. Use Liouville's theorem to conjecture and prove the corresponding theorem for $n = 2$.

3. Evans, p. 86 problem 5

4. Evans, p. 86 problem 6