Partial Differential Equations

Final Exam

December 9, 2004

1. Consider a modified Laplace equation,

$$D \cdot (r\, Du(x)) = 0\,,$$

where $u\colon \mathbb{R}^n \to \mathbb{R}$ for $n \geq 2$, and $r = |x|$.

(a) Show that if u is a radial solution, i.e. $u(x) = v(r)$, then

$$r\, v'' + n\, v' = 0\,.$$

(b) Conclude that any radial solution with $u(r) \to 0$ as $r \to \infty$ is a multiple of

$$u(x) = \frac{1}{r^{n-1}}\,.$$

(5+5)

2. Let $u(x,t) \geq 0$ denote a smooth temperature distribution within a homogeneous, heat-conducting medium U. Hence, u satisfies the heat equation. It is known that the heat energy contained in U is proportional to

$$E = \int_U u(x,t)\, dx\,.$$

(a) Show that the boundary condition $u(x,t) = 0$ for $x \in \partial U$ corresponds to cooling, i.e. that the heat energy is non-increasing.

(b) What boundary condition would you use to describe a perfectly insulated medium?

(c) **Extra credit:** Prove that in case (a), provided U is bounded, the heat energy decreases exponentially.

(5+5+*10*)

3. (a) Let $U \in \mathbb{R}^n$ be open and simply connected, with sufficiently smooth boundary, and assume that the *Poincaré inequality*

$$\|u\|_{L^2} \leq C \|Du\|_{L^2}$$

holds for some constant C and for every $u \in H^1(U)$ with $u = 0$ on ∂U.
Show that

$$C \geq \frac{1}{\sqrt{|\lambda|}} ,$$

where λ is an eigenvalue of the Laplacian corresponding to an eigenfunction that vanishes on ∂U.

(b) Show that there is no Poincaré inequality for $U = \mathbb{R}^n$.

(5+5)

4. Assume that $U \subset \mathbb{R}^n$ is open and connected. Show that if $u \in C^2(U)$ solves the *Neumann problem*

$$\begin{aligned} -\Delta u &= 0 \quad \text{in } U , \\ \nu \cdot Du &= 0 \quad \text{on } \partial U , \end{aligned}$$

then $u = \text{const}$. (10)

5. *Note:* The different parts of this question can be worked on independently. If you get stuck, move on.

(a) Let $u \in H^1([0, 1])$ with

$$u(0) = u(1) = 0 .$$

Prove that

$$|u^m(x)| \leq m \left(\int_0^1 |u|^m \, dx \right)^{\frac{1}{2}} \left(\int_0^1 |u|^{m-2} |u_x|^2 \, dx \right)^{\frac{1}{2}}$$

Hint: Apply the Fundamental Theorem of Calculus to $v = u^m$; Cauchy–Schwarz inequality.

(b) Deduce from (a) that

$$\int_0^1 |u|^{2m} \, dx \leq \frac{m}{2} \left(\int_0^1 |u|^m \, dx \right)^3 + \frac{m}{2} \int_0^1 |u|^{m-2} |u_x|^2 \, dx .$$

(c) Consider the partial differential equation

$$\begin{aligned} u_t &= u_{xx} + u^{m+1} \\ u(0) &= u(1) = 0 \end{aligned}$$

on $U = (0,1) \times (0,T)$, where $m \geq 2$ is an even integer.

Show that

$$\frac{1}{m}\frac{d}{dt}\int_0^1 u^m \, dx = -(m-1)\int_0^1 u^{m-2}\, u_x^2 \, dx + \int_0^1 u^{2m} \, dx \, .$$

(d) Combine (b) and (c) to find that

$$\frac{d}{dt}\int_0^1 u^m \, dx \leq \frac{m^2}{2}\left(\int_0^1 |u|^m \, dx\right)^3 .$$

Finally, conclude that if $u(0) \in L^m([0,1])$, then $u(t) \in L^m([0,1])$ for every $t \in [0,T)$. Give a lower bound for the (potential) blow-up time T.

(5+5+5+5)