General Mathematics and Computational Science I

Exercise 4

September 15, 2005

1. Recall that we defined $\mathbb{Z}$ to be the set of equivalence classes of tuples (a, b) with $a, b \in \mathbb{Z}_+$ with respect to the equivalence relation

   $$(a, b) \sim (a', b') \qquad \text{if and only if} \qquad a + b' = a' + b\,.$$

   Define an order relation by

   $$(a, b) < (c, d) \qquad \text{if and only if} \qquad a + d < b + c\,. \tag{*}$$

   Show that if $(a, b) \sim (a', b')$ and $(a, b) < (c, d)$, then $(a', b') < (c, d)$.

   Remark: This, together with the corresponding statement for the second operand, shows that $\mathbb{Z}$ is well-ordered by relation (*).

2. Recall that addition on $\mathbb{N}$ is the unique binary operation $F\colon \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ with:

   (A1) $F(a, 1) = s(a)$ for all $a \in \mathbb{N}$,
   (A2) $F(a, s(b)) = s(F(a, b))$ for all $a, b \in \mathbb{N}$,

   where $s\colon \mathbb{N} \to \mathbb{N}$ is as in Peano's axioms.

   Prove that F is commutative, i.e. that

   $$F(a, b) = F(b, a) \qquad \text{for all } a, b \in \mathbb{N}.$$

   Hint: Proceed in two steps. First, show that any $a \in \mathbb{N}$ commutes with 1 by proving that

   $$M = \{a \in \mathbb{N}\colon F(a, 1) = F(1, a)\}$$

   is inductive.

3. Multiplication on $\mathbb{N}$ can be defined, similar to addition, as the unique map $G\colon \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ with the following properties:

   (M1) $G(a, 1) = a$ for all $a \in \mathbb{N}$,
   (M2) $G(a, s(b)) = G(a, b) + a$ for all $a, b \in \mathbb{N}$.

   Use this definition to prove that

   $$2 \times 2 = 4$$

   where $2 \equiv s(1)$, $3 \equiv s(2)$, and $4 \equiv s(3)$.