Numerical Methods I – Problem Set 10

Fall Semester 2005

Due December 7 *in class!*

1. The Chebychev polynomial of degree n is given by

$$T_n(x) = \cos(n \arccos x)\,.$$

   (a) Show that the extrema of T_n are located at

$$x_k = \cos\frac{k\pi}{n}$$

   for $k = 0, \ldots, n$, and that the zeros of T_n are located at

$$x_k = \cos\frac{(1+2k)\pi}{2n}$$

   for $k = 0, \ldots, n-1$.

   (b) Show that T_n and T_m are orthogonal with respect to the inner product

$$\langle f, g\rangle = \int_{-1}^{1} \frac{f(x)\,g(x)}{\sqrt{1-x^2}}\,\mathrm{d}x$$

   if and only if $m \neq n$.
   Hint: Use trigonometric substitution, then note that

$$2\,\cos\alpha\,\cos\beta = \cos(\alpha+\beta) + \cos(\alpha-\beta)\,.$$

2. Solve the logistic differential equation

$$\dot{y} = y - y^2\,,$$
$$y(0) = y_0\,.$$

3. Compute the matrix exponential $\exp(A)$ for

   (a) $A = \begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix}$,

(b) $A = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$.

4. Give an example that $\exp(A + B) \neq \exp(A) \exp(B)$.

5. **Project:** Solve the logistic equation from Question 2 on the interval $[0, 1]$ by using the explicit Taylor methods of order 1 and 2. Generate a doubly logarithmic error plot to demonstrate the order of the methods.