Numerical Methods I – Problem Sets 3 and 4

Fall Semester 2005

Due September 30 (Theory), October 7 (Projects)

Background reading: Süli and Mayers, Chapter 2; Quarteroni and Saleri, Chapter 5. Please submit the theoretical homework directly to Stanislav Harizanov (e.g. clearly addressed to him via the mailbox in front of Research I, room 40.)

1. The following root finding method is a modification of the bisection method. It is called *regula falsi*.
$$x_{k+1} = x_0 - \frac{x_k - x_0}{f(x_k) - f(x_0)}\, f(x_0)\,. \tag{*}$$
   (a) Show that the *regula falsi* is consistent.
   (Recall that a method is consistent if every fixed point ξ of this iteration solves the equation $f(\xi) = 0$.)
   (b) Give an argument using Taylor expansion that the *regula falsi* is convergent with order 1.
   (c) The *regula falsi* is applied in the following way. Choose two starting values x_0 and x_1 so that $f(x_0) \cdot f(x_1) < 0$. Compute the sequence of x_k via (*). If $f(x_0) \cdot f(x_{k+1}) > 0$, re-initialize by setting $x_0 := x_k$.
   Show that if f is continuous, the sequence x_k will always converge to a root of f.
   Hint: Notice that x_{k+1} is the zero of the line joining the points $(x_0, f(x_0))$ and $(x_k, f(x_k))$. Then note that the sequence of intervals that bracket the root has monotonic bounds.

2. (From SM.) Recall that if $\|\boldsymbol{x}\|_p$ denotes the p-norm of a vector $\boldsymbol{x} \in \mathbb{R}^n$, then the associated norm for a matrix $A \in \mathbb{R}^{n\times n}$ is defined
$$\|A\|_p = \max_{\boldsymbol{x}\neq 0} \frac{\|A\boldsymbol{x}\|_p}{\|\boldsymbol{x}\|_p}\,.$$
   Suppose that for a matrix $A \in \mathbb{R}^{n\times n}$,
$$\sum_{i=1}^{n} |a_{ij}| \le C \tag{1}$$
   for $j = 1, \ldots, n$.

(a) Show that, for any vector $\boldsymbol{x} \in \mathbb{R}^n$,

$$\|A\boldsymbol{x}\|_1 \leq C\,\|\boldsymbol{x}\|_1 \,. \tag{2}$$

(b) Find C subject to (1) and a nonzero vector $\boldsymbol{x}$ so that (2) holds with equality.

(c) Conclude that

$$\|A\|_1 = \max_{j=1,\ldots,n} \sum_{i=1}^{n} |a_{ij}| \,.$$

3. (a) Show that, for $A \in \mathbb{R}^{n\times n}$,

$$\|A\|_2 = \sqrt{\lambda_{\max}}\,,$$

where $\lambda_{\max}$ is the largest eigenvalue of $A^T A$.
Hint: Recall that for any $\boldsymbol{x} \in \mathbb{R}^n$, you can write $\boldsymbol{x}^T\boldsymbol{x} = \|\boldsymbol{x}\|_2$. The eigenvalues of the symmetric matrix $A^T A$ are real and nonnegative (why?), and its eigenvectors can be chosen to form an orthonormal basis.

(b) Conclude that the condition number of an orthogonal matrix is 1.

4. Let $\rho(A)$ denote the spectral radius of a matrix A, i.e.

$$\rho(A) = \max_{i=1,\ldots,m} |\lambda_i|\,,$$

where $\lambda_1, \ldots, \lambda_m$ are the (possibly complex) eigenvalues of A.

(a) Show that $\rho(A) \leq \|A\|$ for any matrix norm of A.

(b) Give an example of a nonzero matrix for which $\rho(A) = 0$.

5. (From SM.) Assume that a nonsingular matrix $A \in \mathbb{R}^{n\times n}$ has an LU factorization. Show that A can also be factored in the form $A = LDR$ where L is unit lower triangular, D is diagonal, and R is unit upper triangular. Use this result to express the LU factoriztion of A^T in terms of the LU factorization of A.

6. The $n \times n$ Vandermonde Matrix is defined as

$$V = \begin{pmatrix} 1 & 2 & 4 & 8 & \cdots & 2^{n-1} \\ 1 & 3 & 9 & 27 & \cdots & 3^{n-1} \\ 1 & 4 & 16 & 64 & \cdots & 4^{n-1} \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ 1 & n+1 & (n+1)^2 & (n+1)^3 & \cdots & (n+1)^{n-1} \end{pmatrix}.$$

Let $\boldsymbol{b} \in \mathbb{R}^n$ be the vector containing the sum of each row of V. Find a formula for the components of $\boldsymbol{b}$. What is the solution of the system of linear equations $V\boldsymbol{x} = \boldsymbol{b}$?

7. **Project:** Write an Octave function `vandermonde(n)` that generates the $n \times n$ Vandermonde Matrix. Compute its condition number with respect to the 2-norm for different values of n. (You may use the built-in Octave function `cond`.)

8. **Project:** Write an Octave function that computes the LU decomposition of a matrix without pivoting. Test your code by solving $V\boldsymbol{x} = \boldsymbol{b}$ from Question 4 for several values of n.

9. **Project:** Modify your program for the LU decomposition to include pivoting. I.e., let your program find P, L, and R so that for a given nonsingular matrix $A \in \mathbb{R}^{n \times n}$, $PA = LU$. Use your program to solve last week's Vandermonde test problem. Does pivoting help?

10. **Project:** Use LU decomposition with and without pivoting to solve randomly generated linear equations. Print the 2-norm of the residual and the condition number of the matrix for many test problems. Describe the effect of pivoting.

11. **Project:** Write an Octave code to perform a QR decomposition of an $n \times m$ matrix. Then compute the least squares solution to $A\boldsymbol{x} = \boldsymbol{b}$ with

$$A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}, \qquad \boldsymbol{b} = \begin{pmatrix} 4 \\ 3 \\ 2 \\ 1 \end{pmatrix}.$$