Numerical Methods I – Problem Set 5

Fall Semester 2005

Due October 14

1. Consider the following class of iterative methods for solving $A\boldsymbol{x} = \boldsymbol{b}$, called *stationary Richardson methods*:

$$\boldsymbol{x}_{k+1} = \boldsymbol{x}_k + \alpha\, P^{-1}\boldsymbol{r}_k$$

where $\boldsymbol{r}_k = \boldsymbol{b} - A\boldsymbol{x}_k$ is the residual in the kth step. Assume that all eigenvalues of $P^{-1}A$ are real and positive, and let $\lambda_{\min}$ and $\lambda_{\max}$ denote the minimum and maximum eigenvalue, respectively.

   (a) Show that the method converges if and only if $\alpha < 2/\lambda_{\max}$.

   (b) Show that the method converges fastest if

$$\alpha = \frac{2}{\lambda_{\min} + \lambda_{\max}}\,.$$

2. Given any iterative method for solving $A\boldsymbol{x} = \boldsymbol{b}$ of the form

$$\boldsymbol{x}_{k+1} = \boldsymbol{x}_k + B\boldsymbol{r}_k$$

where, again, $\boldsymbol{r}_k = \boldsymbol{b} - A\boldsymbol{x}_k$. Show that the method is consistent for any nonsingular B.

(Recall: a method is consistent if any fixed point of the method is a solution to $A\boldsymbol{x} = \boldsymbol{b}$.)

3. A matrix A is called *strictly diagonally dominant by rows* if

$$|a_{ii}| > \sum_{\substack{j=1 \\ j\neq i}}^{n} |a_{ij}|$$

for every $i = 1, \ldots, n$.

Show that if A has this property, then the Jacobi method converges.

4. **Project:** Write an Octave code to solve $A\boldsymbol{x} = \boldsymbol{b}$ by each of the the following methods:

(a) Jacobi method

(b) Gradient method

Plot the 2-norm of the residual on a logarithmic scale vs. the number of iterations for the following $n \times n$ test problem for $n = 10$:

$$A = \begin{pmatrix} 2 & -1 & 0 & & \cdots & & 0 & -\frac{1}{2} \\ -1 & 2 & \ddots & \ddots & & & & 0 \\ 0 & \ddots & \ddots & & & & & \\ & \ddots & & & & & & \vdots \\ \vdots & & & & & & \ddots & \\ & & & & & \ddots & \ddots & 0 \\ 0 & & & & \ddots & \ddots & 2 & -1 \\ -\frac{1}{2} & 0 & & \cdots & & 0 & -1 & 2 \end{pmatrix}, \qquad \boldsymbol{b} = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}.$$

Which method works best?