Numerical Methods I – Problem Set 6

Fall Semester 2005

Due October 21

1. It can be shown that the Conjugate Gradient (CG) method in the absence of rounding errors terminates after at most n steps, yielding the exact solution. Compare the number of operations for solving a system of linear equations via LU decomposition with the number of operations for n iterations of CG.

   Remark: Since A must be symmetric and positive definite for CG to be applicable, it is actually possible to half the number of operations required for the LU decomposition by using the so-called Cholesky decomposition. However, this does not change the overall picture. For background information, see SM, pp. 90–93.

2. **Project:** Modify your Octave code from Homework 5 to use the Conjugate Gradient rather than the simple Gradient method. How do the two methods compare for the given test problem?

3. (a) Compute the Lagrange polynomial which interpolates a function f at three distinct nodes x_0, x_1, and x_2.
   (b) Use the result from part (a) to derive approximations for $f'(x_1)$ and $f''(x_1)$.
   (c) Simplify the formulas from (b) for the case of equidistant nodes.

4. (From SM.) Given a set of points

   $$(x_0, y_0), (x_1, y_1), \ldots, (x_{n+1}, y_{n+1})$$

   with distinct $x_1, \ldots, x_{n+1}$, let q be the Lagrange polynomial of degree n interpolating the points with index $i = 0, \ldots, n$, and let r be the Lagrange polynomial of degree n interpolating the points with index $i = 1, \ldots, n+1$. Show that the Lagrange polynomial of degree $n + 1$ interpolating all $n + 2$ points is given by

   $$p(x) = \frac{(x - x_0)\, r(x) - (x - x_{n+1})\, q(x)}{x_{n+1} - x_0}\,.$$

5. **Project:** (Runge's example.) Apply Lagrange interpolation with n equidistant interpolation points on the interval $[-5, 5]$ to the function

   $$f(x) = \frac{1}{1 + x^2}\,.$$

   Plot f and the Lagrange polynomial p_n for different values of n. How does the error behave as n increases?