Numerical Methods I – Lab 4

Fall Semester 2005

October 11, 2005

Iterative refinement is a technique for cheaply improving the accuracy of the solution of $A\boldsymbol{x} = \boldsymbol{b}$ initially obtained through LU-decomposition. It was widely used in times when computations where usually single precision, and the accuracy of the obtained solutions was often insufficient. It is sometimes useful even today.

The Algorithm is as follows.

1. (Usual LU-solve.) Given $A\boldsymbol{x} = \boldsymbol{b}$ with nonsingular $A \in M(n \times n)$, find the LU-factorization of A. Solve $L\boldsymbol{y} = \boldsymbol{b}$ (forward substitution), then $U\boldsymbol{x} = \boldsymbol{y}$ (backward substitution).

2. (Compute residual.) Compute $\boldsymbol{r} = \boldsymbol{b} - A\boldsymbol{x}$, called the residual. If possible, this step should be done in higher precision.

3. (Update.) Compute $\boldsymbol{x}_{\text{new}} = \boldsymbol{x} + \boldsymbol{z}$, where $\boldsymbol{z}$ solves $A\boldsymbol{z} = \boldsymbol{r}$; use the LU-factorization from part (1) to solve this system.

Steps (2) and (3) can be repeated if necessary.

Tasks:

1. Write a program that implements the above algorithm. For obtaining the LU-factorization, you may use the *Octave* function `lu`. In order to mimic single precision, perturb the entries of U and the off-diagonal entries of L by random small relative error $\leq 10^{-7}$. To mimic additional stability problems, larger perturbations may be tried as well.

2. Test the effect of iterative refinement on the Vandermonde test problem from the homework. Give an estimate of the number of operations in Step 2 and 3, and compare it with the operation count for Step 1.

3. You may interpret iterative refinement as a linear iterative method for solving $A\boldsymbol{x} = \boldsymbol{b}$ with the starting vector $\boldsymbol{x}_0$ obtained in Step 1, or even with $\boldsymbol{x}_0 = 0$. What is its iteration matrix B? Can you explain (based on your knowledge on the convergence theory of linear iterative methods) why in practice one step of iterative refinement is often sufficient to produce a high-accuracy result? This question relates to the testing experience in part (2).