General Mathematics and Computational Science I

Exercise 3

September 21, 2006

1. Recall that we defined $\mathbb{Z}$ to be the set of equivalence classes of tuples (a, b) with $a, b \in \mathbb{Z}_+$ with respect to the equivalence relation

$$(a, b) \sim (a', b') \qquad \text{if and only if} \qquad a + b' = a' + b\,.$$

Define an order relation by

$$(a, b) < (c, d) \qquad \text{if and only if} \qquad a + d < b + c\,. \tag{*}$$

Show that if $(a, b) \sim (a', b')$ and $(a, b) < (c, d)$, then $(a', b') < (c, d)$.

Remark: This, together with the corresponding statement for the second operand, shows that $\mathbb{Z}$ is well-ordered by relation (*).

2. Check if the following functions are injective, surjective or bijective.

   (i) $f\colon \mathbb{N} \to \mathbb{N}$, given by $f(n) = n + 1$.

   (ii) $f\colon \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$, given by $f(n, m) = \min\{m, n\}$.

   (iii) $f\colon \mathbb{N} \to \mathbb{N}$ with $f(2n) = 2n - 1$ for all $n > 0$ and $f(2n+1) = 2n+2$ for all $n \geq 0$.

   (If one of the properties fails, give a counter example. Otherwise give a short proof.)

3. Multiplication on $\mathbb{N}$ can be defined, similar to addition, as the unique map $G\colon \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ with the following properties:

(M1) $G(a, 1) = a$ for all $a \in \mathbb{N}$,

(M2) $G(a, s(b)) = G(a, b) + a$ for all $a, b \in \mathbb{N}$.

Use this definition to prove that

$$2 \times 2 = 4$$

where $2 \equiv s(1)$, $3 \equiv s(2)$, and $4 \equiv s(3)$.