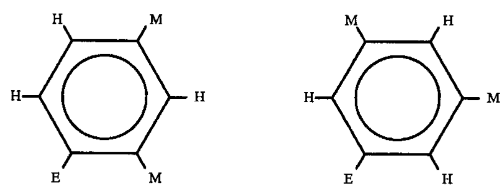
1. In chemistry, benzene is a molecule made of six carbon atoms forming a hexagon, each of which also bonds to a hydrogen atom (H). In how many ways can hydrogen atoms be substituted by two methyl groups (M) and one ethyl group (E) as to form distinct molecules?

   As an example, two distinct configurations are shown:



(8)

Note that two molecules are identical if one can be transformed into the other by

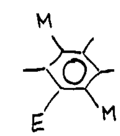
- rotation in the plane
- flipping the molecule upside-down (Chemistry is three-dimensional!)

Solution 1:

- Fix the location of E (the molecule can always be rotated such that E is in a fixed location)
- Flip the molecule such that at least one of the two clockwise adjacent sites of E is occupied by M (this can always be done as there are only 5 sites available for Ms

Case 1: The clockwise adjacent site to E is occupied by M. Then there are four possibilities to place the second M.

Case 2: The clockwise adjacent site to E is occupied by H. Then the next site must be M and there are two possibilities

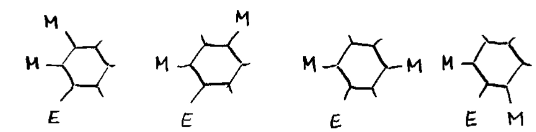


is a mirror version of a configuration already counted in case 1, and must not be counted again.)

So there are $4+2=6$ distinct configurations.

(It's actually easy to list them all;

Case 1:



Case 2:

)

Solution 2: The number of distinct configurations is

$$\underbrace{\frac{6!}{3!\,2!\,1!}}_{\text{number of possibilities with 6 distinct sites}} \frac{1}{6}\,\frac{1}{2} + \underbrace{\frac{1}{2}\cdot 2}_{} = \frac{5\cdot 4\cdot 3}{3\cdot 4} + 1 = 6$$

rotation does not result in new configuration

but we have two symmetric configurations which we should not have included in the division by 2. So we have to add half of this number back in.

flipping about axis through E does not result in new configuration

2. Prove the so-called *Bernoulli inequality*

$$(1+x)^n \geq 1 + nx$$

for any real number $x > -1$ and $n \in \mathbb{N}$. (8)

Proof 1:

By induction:

$n=1$: $(1+x)^1 = 1 + 1 \cdot x$

$n \to n+1$: $(1+x)^{n+1} = (1+x)^n (1+x)$

$\overset{\text{I.H.}}{\geq} (1+nx)(1+x)$ (permitted since $1+x>0$!)

$= 1 + nx + x + nx^2$

$\geq 1 + (n+1)x$ $\square$

Remark: For $x \geq 0$, the following proof is also possible:

$$(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k \geq \binom{n}{0} x^0 + \binom{n}{1} x^1 = 1 + nx .$$

Proof 2: If $-1 < x$ and $1+nx<0$, then LHS is positive and RHS is negative, so the inequality is obviously true.

Let therefore $1+nx \geq 0$. Then

$$\frac{(1+nx) + \overbrace{1+\ldots+1}^{(n-1)\text{ terms}}}{n} \overset{\text{AM-GM}}{\geq} \Big((1+nx) \cdot \underbrace{1 \cdot \ldots \cdot 1}_{(n-1)\text{ times}}\Big)^{\frac{1}{n}}$$

$$\begin{array}{ccc} \| & & \| \\ 1+x & & (1+nx)^{\frac{1}{n}} \end{array}$$

$\square$

3. (a) You flip a coin five times in a row. What is the probability that it comes up heads five times?

(b) You flip a coin ten times in a row. Which of the following outcomes is more likely? HHHHHHHHHH or THTTHHTHHT? Explain!

(6+4)

(a) $\frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{2^5} = \frac{1}{32}$

(b) Provided the coin is unbiased, each is just one particular 10-member sequence consisting of Hs and Ts, so just one particular outcome from the sample space, and therefore equally likely.

4. Use the method of generating functions to find a closed form expression for the members of the sequence

$$a_{n+1} = 2\,a_{n-1} - a_n$$

where $a_0 = 1$ and $a_1 = -2$. (8)

Set $\quad \phi(x) = \sum_{i=0}^{\infty} a_i x^i$

$$\Rightarrow \quad (1+x-2x^2)\,\phi(x) = a_0 + a_1 x + a_2 x^2 + \dots + a_0 x + a_1 x^2 + \dots \underbrace{- 2a_0 x^2 - \dots}_{=0}$$

$$= 1 - x$$

$$\Rightarrow \quad \phi(x) = \frac{1-x}{1+x-2x^2} = \frac{1-x}{(1-x)(1+2x)} = \frac{1}{1+2x}$$

$$= \sum_{i=0}^{\infty} (-2x)^i$$

$$\Rightarrow \quad a_n = (-2)^n$$

5. Find the largest value of the function $f(x,y) = xy$ on the set of points satisfying the inequality $\underbrace{x^2 + xy + y^2 \leq 1}_{(*)}$. (8)

$$xy \leq \frac{1}{2}x^2 + \frac{1}{2}y^2$$

$$\leq \frac{1}{2}(1 - xy)$$

$$\Rightarrow \quad \frac{3}{2}xy \leq \frac{1}{2}$$

$$\Rightarrow \quad xy \leq \frac{1}{3}$$

with equality iff $x = y$.

So the largest value of $f(x,y) = xy$ on the given set is indeed $\frac{1}{3}$.

6. Show that

$$(a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right) \geq 9$$

for arbitrary positive real numbers a, b and c. When does equality hold? (8)

Use the Cauchy inequality:

$$3 = \sqrt{a}\,\frac{1}{\sqrt{a}} + \sqrt{b}\,\frac{1}{\sqrt{b}} + \sqrt{c}\,\frac{1}{\sqrt{c}}$$

$$\leq \sqrt{a+b+c}\,\sqrt{\frac{1}{a}+\frac{1}{b}+\frac{1}{c}}$$

Now take the square. $\square$