Engineering and Science Mathematics 2B

Final Exam

May 24, 2003

For questions with an easy (E) and an advanced (A) version, choose either the easy or the advanced version, not both. You may make a different choice for each question.

1. (a) Compute the inverse of the Matrix

$$A = \begin{pmatrix} i & 0 & -i \\ i & -i & 0 \\ 0 & i & i \end{pmatrix} .$$

   (b) Using the result from part (a), or otherwise, solve the system of linear equations

$$A\,\boldsymbol{x} = \begin{pmatrix} i \\ 1 \\ -i \end{pmatrix} .$$

(10+5)

2. (E) Find the matrix representing—with respect to the standard basis—the projection in $\mathbb{R}^2$ onto the vector

$$\boldsymbol{v} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} .$$

(4)

   (A) Let V be a real vector space with an inner product, $\{\boldsymbol{e}_1, \ldots, \boldsymbol{e}_n\}$ an orthonormal basis for V, and T a linear transformation on V. Explain why the matrix S which represents T with respect to the basis $\{\boldsymbol{e}_1, \ldots, \boldsymbol{e}_n\}$ has the components

$$s_{ij} = \langle T(\boldsymbol{e}_j), \boldsymbol{e}_i \rangle .$$

(5)

3. Consider the vector space P_2 of polynomials of degree less or equal to 2 with basis

$$B = \{1, x, x^2\} .$$

   (a) What is the matrix S representing the derivative on P_2 with respect to the basis B?

(b) Find the eigenvalues and eigenvectors of S.

(c) Show that S is not diagonalizable. Is there another basis for P_2 such that the derivative with respect to this other basis is diagonalizable? Explain.

(5+5+5)

4. Compute the complex Fourier series for the function $f(x) = x$ on the interval $[-\pi, \pi]$. (10)

5. (E) Show that

$$\mathcal{F}\left(\frac{df}{dx}\right) = i\xi\, \mathcal{F}(f)\,,$$

where $\mathcal{F}$ denotes the Fourier transform. (8)

(A) It is known that the Fourier transform of a Gaussian distribution is again a (non-normalized) Gaussian distribution. More precisely, if ϕ_σ denotes the Gaussian with mean zero and variance σ^2, i.e.

$$\phi_\sigma(x) = \frac{1}{\sigma\sqrt{2\pi}}\, e^{-\frac{x^2}{2\sigma^2}}\,,$$

then

$$\tilde{\phi}_\sigma(\xi) = \frac{1}{\sqrt{2\pi}}\, e^{-\frac{\sigma^2\xi^2}{2}}\,.$$

Show that the convolution of two Gaussian distributions ϕ_σ and ϕ_μ is again a Gaussian. What is the variance of the resulting Gaussian? (10)

6. (E) Compute

$$\int_{-\infty}^{\infty} \delta(x^{1/3})\, dx\,.$$

(8)

(A) If X is a continuous random variable with probability density function $p(x)$, and $Y = g(X)$ another random variable, then the probability density function of Y is given by

$$q(y) = \int_{-\infty}^{\infty} p(x)\, \delta(g(x) - y)\, dx\,.$$

Compute $q(y)$ when $g(x) = x^{1/3}$, and $p(x) = 1$ on the interval $[0, 1]$ and $p(x) = 0$ otherwise.

(10)

7. Each question on a multiple choice exam has three possible answers, of which only one is correct. Some student did not study very hard and there is only a 50% chance that he knows the anwer to a particular question. Of course he will select an answer at random in case he does not know the correct one. What is the probability that he knows the answer to a question he has answered correctly? (10)

8. A homework set in ESM 2B has six questions. For the corresponding quiz, three of these questions are selected at random, each selection being independent of the others.

   (E) What is the probability that you and your best friend get an identical quiz? (8)

   (A) Find a formula for the probability that in a group of k students at least two get an identical quiz. (10)

9. An experiment is independently repeated n times. The experiment has two possible outcomes: "success" with probability p and "failure" with probability $q = 1 - p$. Consider the random variable

$$X = \text{number of trials required to obtain the first success}\,.$$

   (a) Explain why the probability function of X is

$$P(X = k) = p\, q^{k-1}\,.$$

   (b) Show that the moment generating function $M_X(t) = E[e^{tX}]$ is given by

$$M_X(t) = \frac{p\, e^t}{1 - q\, e^t}\,.$$

   (c) Compute, by using the moment generating function, or otherwise, the mean and the variance of X.

(5+5+5)