Problem 5 of homework 6:

An orthonormal basis is $\left\{c_0 = \frac{1}{\sqrt{L}}, c_k = \sqrt{\frac{2}{L}}\cos\left(\frac{2\pi kx}{L}\right), s_k = \sqrt{\frac{2}{L}}\sin\left(\frac{2\pi kx}{L}\right); k \in \mathbb{N}\right\}$

$$f(x) = \langle f(x), c_0\rangle c_0 + \sum_{k=1}^{\infty}\langle f(x), c_k\rangle c_k + \sum_{k=1}^{\infty}\langle f(x), s_k\rangle s_k$$

Here $f(x) = (x+\pi)^2$ on the intervall $[-\pi, \pi)$, so $L = 2\pi$.

$$\begin{aligned}
\langle f(x), c_0\rangle &= \frac{1}{\sqrt{2\pi}}\int_{-\pi}^{\pi}(x+\pi)^2\,dx = \frac{1}{\sqrt{2\pi}}\,\frac{1}{3}\,(x+\pi)^3\Big|_{-\pi}^{\pi} = \frac{8\pi^3}{3\sqrt{2\pi}}\\
\langle f(x), c_k\rangle &= \frac{1}{\sqrt{\pi}}\int_{-\pi}^{\pi}(x+\pi)^2\cos(kx)\,dx = \frac{1}{\sqrt{\pi}}\left((x+\pi)^2\frac{1}{k}\,\sin(kx)\Big|_{-\pi}^{\pi} - \int_{-\pi}^{\pi}2(x+\pi)\frac{1}{k}\sin(kx)\,dx\right)\\
&= \frac{1}{\sqrt{\pi}}\left(0 - 2\left((x+\pi)\frac{-1}{k^2}\,\cos(kx)\Big|_{-\pi}^{\pi} + \int_{-\pi}^{\pi}\frac{1}{k^2}\cos(kx)\,dx\right)\right)\\
&= \frac{2}{\sqrt{\pi}}\left(2\pi\frac{(-1)^k}{k^2} - 0\right) = \frac{4\pi}{\sqrt{\pi}}\,\frac{(-1)^k}{k^2}\\
\langle f(x), s_k\rangle &= \frac{1}{\sqrt{\pi}}\int_{-\pi}^{\pi}(x+\pi)^2\sin(kx)\,dx = \frac{1}{\sqrt{\pi}}\left((x+\pi)^2\frac{-1}{k}\,\cos(kx)\Big|_{-\pi}^{\pi} + \int_{-\pi}^{\pi}2(x+\pi)\frac{1}{k}\cos(kx)\,dx\right)\\
&= \frac{1}{\sqrt{\pi}}\left(-4\pi^2\frac{(-1)^k}{k} + 2\left((x+\pi)\frac{1}{k^2}\,\sin(kx)\Big|_{-\pi}^{\pi} - \int_{-\pi}^{\pi}\frac{1}{k^2}\sin(kx)\,dx\right)\right)\\
&= \frac{1}{\sqrt{\pi}}\left(-4\pi^2\frac{(-1)^k}{k} + 0 - 0\right) = -\frac{4\pi^2}{\sqrt{\pi}}\,\frac{(-1)^k}{k}
\end{aligned}$$

So the fourier representation of f is:

$$\begin{aligned}
f(x) &= \frac{8}{3\sqrt{2\pi}}\,\pi^3\cdot\frac{1}{\sqrt{2\pi}} + \frac{4\pi}{\sqrt{\pi}}\sum_{k=1}^{\infty}\frac{(-1)^k}{k^2}\cdot\frac{1}{\sqrt{\pi}}\cos kx - \frac{4\pi^2}{\sqrt{\pi}}\sum_{k=1}^{\infty}\frac{(-1)^k}{k}\cdot\frac{1}{\sqrt{\pi}}\sin kx\\
&= \frac{4}{3}\,\pi^2 + 4\sum_{k=1}^{\infty}\frac{(-1)^k}{k^2}\cos kx - 4\pi\sum_{k=1}^{\infty}\frac{(-1)^k}{k}\sin kx
\end{aligned}$$

The values to which the Fourier series converges at $-\pi$ and π are the same, because of the period 2π. For $x = \pi$ or $-\pi$ we get

$$\frac{4}{3}\,\pi^2 + 4\sum_{k=1}^{\infty}\frac{(-1)^k}{k^2}\cos k\pi - 4\pi\sum_{k=1}^{\infty}\frac{(-1)^k}{k}\sin k\pi = \frac{4}{3}\,\pi^2 + 4\sum_{k=1}^{\infty}\frac{1}{k^2} = \frac{4}{3}\,\pi^2 + 4\frac{\pi^2}{6} = 2\pi^2\,,$$

for $x = 0$ or 2π we get

$$\frac{4}{3}\,\pi^2 + 4\sum_{k=1}^{\infty}\frac{(-1)^k}{k^2}\,,$$

which has to be the same as $f(0) = \pi^2$, because $0 \in (-\pi, \pi)$.