Engineering and Science Mathematics 2B

Midterm I

March 5, 2003

For questions with an easy (E) and an advanced (A) version, choose either the easy or the advanced version, not both. You may make a different choice for each question.

1. (a) Find the normal equation for the plane that passes through the points $(0,-1,1)$, $(2,2,-1)$, and $(1,1,3)$.
   (b) Find the distance of the point $(1,1,1)$ to this plane.

   (5+5)

2. Find the general solution to the system of linear equations $A\boldsymbol{x} = \boldsymbol{b}$ with

$$A = \begin{pmatrix} 1 & -2 & 1 & 1 \\ 2 & -4 & 2 & -2 \\ 2 & -4 & 2 & 6 \end{pmatrix}, \qquad \boldsymbol{b} = \begin{pmatrix} 1 \\ -2 \\ 6 \end{pmatrix}$$

   Check your answer! (10)

3. (E) Consider the matrix A from the previous question. Show that $A\boldsymbol{x} = \boldsymbol{c}$ does not have a solution when

$$\boldsymbol{c} = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}.$$

   (4)

   (A) Consider the matrix A from the previous question. Characterize all vectors $\boldsymbol{c} \in \mathbb{R}^3$ such that $A\boldsymbol{x} = \boldsymbol{c}$ is solvable. (5)

4. (E) Decide whether the following set of vectors is a basis of $\mathbb{R}^3$. Justify your answer.

$$\boldsymbol{v}_1 = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}, \quad \boldsymbol{v}_2 = \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix}, \quad \boldsymbol{v}_3 = \begin{pmatrix} -1 \\ 1 \\ -2 \end{pmatrix}.$$

   (4)

   (A) Prove that the columns of any $n \times k$ matrix with $n < k$ are linearly dependent.

   (5)

5. (E) Let $A = \begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{pmatrix}$.

   (a) Find the eigenvalues and eigenvectors of A.

   (b) Write out a diagonal matrix D and an invertible matrix S such that $D = S^{-1}AS$.

   (c) Check your result by explicitly performing the matrix multiplications SD and AS.

   (8+4+4)

   (A) (a) Find the matrix A representing the linear transformation on $\mathbb{R}^2$ that maps the triangle with vertices $\boldsymbol{a} = (0,0)$, $\boldsymbol{b} = (1,0)$, $\boldsymbol{c} = (1,1)$ onto the triangle with vertices $\boldsymbol{a}' = (0,0)$, $\boldsymbol{b}' = (\frac{1}{2}, \frac{3}{2})$, $\boldsymbol{c}' = (2,2)$.

   (b) Diagonalize A. I.e., find a diagonal matrix D and a change of coordinate S such that $D = S^{-1}AS$.

   (c) Use the result from (b) to give a concise geometric description of the linear transformation.

   (5+10+5)

6. Let V be the vector space of continuous functions on $\mathbb{R}$ spanned by the basis

$$E = \{e^{ix}, e^{-ix}\}$$

with the usual addition and scalar multiplication.

   (a) Find the matrix D which represents the derivative operator on V with respect to the basis E.

   (b) Let

$$T = \{\sin x, \cos x\}.$$

   Explain why T is also a basis of V, and find the matrix S for the change of basis from T to E.
   (Recall that $e^{ix} = \cos x + i \sin x$.)

   (c) Find the matrix C which represents the derivative with respect to the new basis T.

   (5+5+5)

7. (E) If $A \in M(n \times n)$, show that $B = A^T A$ is a symmetric matrix.
   (Recall that a matrix C is symmetric if $C = C^T$.) (4)

   (A) Prove that the eigenvalues of any real symmetric matrix are real. (5)