

Engineering and Science Mathematics 2B

Homework 7

due March 29, 2003, before 22:00

Normal questions and advanced questions (A) are worth 5 points; easy questions (E) are worth 4 points. Complete either the easy, or the advanced version, not both.

1. Let $f(t) = \sin t$ on the interval $[0, \frac{\pi}{2})$, and periodically extended outside of this interval, be represented by the complex Fourier series

$$\sum_{n=-\infty}^{\infty} c_n \frac{e^{Anti}}{\sqrt{\frac{\pi}{2}}}.$$

Compute the Fourier coefficients c_n .

2. Find the Fourier transform of $f(x) = e^{-|x|}$.
3. Show that $\mathcal{F}(f(x+a)) = e^{ia\xi} \mathcal{F}(f)$.
4. (E) Show that $\mathcal{F}(f') = i\xi \mathcal{F}(f)$.

Hint: Integration by parts. You may assume that all boundary terms are zero when integrating by parts.

- (A) By taking the Fourier transform of the equation

$$\frac{d^2 u}{dx^2} - u = f,$$

show that the solution $u(x)$ can be written as

$$u(x) = \frac{-1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{e^{i\xi x} \tilde{f}(\xi)}{1 + \xi^2} d\xi,$$

where $\tilde{f}(\xi)$ is the Fourier transform of $f(x)$.

5. Compute the integral

$$(E) \int_{-2}^2 \delta(2x) \cos x dx,$$

(A) $\int_{-2\pi}^{2\pi} \delta(x^2 - \pi^2) \cos x \, dx.$

6. (E) Let the *Heavyside* or *unit step function* be defined by

$$H(x) = \begin{cases} 0 & \text{for } x < 0 \\ \frac{1}{2} & \text{for } x = 0 \\ 1 & \text{for } x > 0. \end{cases}$$

Show that $H'(x) = \delta(x)$.

Hint: Integration by parts.

(A) One can define (so-called distributional) derivatives of the δ -function via

$$\int_{-\infty}^{\infty} f(x) \delta^{(n)}(x) \, dx = (-1)^n f^{(n)}(0)$$

for any n times differentiable function $f(x)$.

So we may be tempted to write a Taylor series for the δ -function,

$$\delta(x + a) = \sum_{n=0}^{\infty} \frac{\delta^{(n)}(x)}{n!} a^n.$$

It appears that the left side is zero except at $x = -a$, while the right side is zero except at $x = 0$. Resolve this paradox.