

# Engineering and Science Mathematics 2B

## Homework 9

due May 10, 2003, before 22:00

Normal questions and advanced questions (A) are worth 5 points; easy questions (E) are worth 4 points. Complete either the easy, or the advanced version, not both.

1. Let  $A$  and  $B$  be two *statistically independent* events. Suppose  $P(A) = \frac{1}{3}$  and  $P(B) = \frac{1}{4}$ . Compute the probabilities  $P(A|B)$ ,  $P(B|A)$ ,  $P(A \cup B)$ ,  $P(A \cap B)$ ,  $P(A - B)$ , and  $P(B - A)$ .
2. A boy is selected at random from among the children belonging to families with  $n$  children.
  - (E) What is the probability that the boy has  $k - 1$  brothers?
  - (A) It is known that the boy has at least two sisters. Show that the probability that he has  $k - 1$  brothers is

$$\frac{(n-1)!}{(2^{n-1} - n)(k-1)!(n-k)!}$$

when  $1 \leq k \leq n - 2$ , and zero for other values of  $k$ .

Hint: Use part (E) and Bayes' rule.

3. Gamblers  $A$  and  $B$  each have two unbiased four-sided dice, the four faces being numbered 1, 2, 3, 4. Without looking,  $B$  tries to guess the sum  $x$  of the numbers on the bottom faces of  $A$ 's two dice after they have been thrown onto a table. If the guess is correct,  $B$  receives  $x^2$  Euros, but if not he loses  $x$  Euros.
  - (E) Show that, when guessing the sum of  $x$ ,  $B$ 's expected gain  $G$  per throw of  $A$ 's dice is
$$E[G_x] = p_x(x^2 + x) - x,$$
where  $p_x$  is the probability that the sum of the bottom faces is  $x$ .
  - (A) Compute the expected gain of  $B$  if he always guesses the sum of  $A$ 's bottom faces from the previous round.
4. In how many ways can 8 people be placed around a table if there are three who insist on sitting together?

5. Prove the following identities:

$$(E) \quad {}^n C_k \quad {}^k C_\ell = {}^n C_\ell \quad {}^{n-\ell} C_{k-\ell}$$

$$(A) \quad \sum_{i=0}^k {}^m C_i \quad {}^n C_{k-i} = {}^{m+n} C_k$$

6. A royal family has children until it has a boy or until it has three children, whichever comes first. Assume that each child is a boy with probability  $\frac{1}{2}$ . Find the expected number of boys in this family and the expected number of girls.