Engineering and Science Mathematics 2B

Midterm I

March 3, 2004

1. (E) Find the distance of the point $\boldsymbol{p} = (0,1,0)^T$ to the plane

$$(\boldsymbol{x} - \boldsymbol{a}) \cdot \boldsymbol{n} = 0$$

where $\boldsymbol{a} = (1,0,1)^T$ and $\boldsymbol{n} = \frac{1}{\sqrt{3}}(1,-1,-1)^T$.

(8)

(A) Show that two lines in $\mathbb{R}^3$ lie in a plane only if either they intersect or they are parallel. (10)

2. Find the general solution to the system of linear equations $A\boldsymbol{x} = \boldsymbol{b}$ with

$$A = \begin{pmatrix} 0 & 1 & 2 & 3 \\ -1 & -2 & -3 & -4 \\ 1 & 1 & 1 & 1 \end{pmatrix}, \qquad \boldsymbol{b} = \begin{pmatrix} 6 \\ -10 \\ 4 \end{pmatrix}$$

Check your answer! (10)

3. (E) Consider the matrix A from the previous question. Show that $A\boldsymbol{x} = \boldsymbol{c}$ does not have a solution when

$$\boldsymbol{c} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

(8)

(A) Consider the matrix A from the previous question. Characterize all vectors $\boldsymbol{c} \in \mathbb{R}^3$ such that $A\boldsymbol{x} = \boldsymbol{c}$ is solvable. (10)

4. Let $A = \begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix}$.

   (a) Find the eigenvalues and eigenvectors of A.

   (b) Write out a diagonal matrix D and an invertible matrix S such that $D = S^{-1}AS$.

   (c) Check your result by explicitly performing the matrix multiplications SD and AS.

(10+5+5)

5. (E) Give an example of two matrices A and B such that $AB \neq BA$. (An explicit calculation is required!) (8)

   (A) Let $A, B \in M(n \times n)$ such that there exists a basis $\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n\}$ of $\mathbb{R}^n$ that is also a set of eigenvectors for both A and B. Show that, in this case, $AB = BA$.

(10)

6. (E) Let

$$\boldsymbol{v}_1 = \begin{pmatrix} 2 \\ -2 \\ -1 \end{pmatrix}, \quad \boldsymbol{v}_2 = \begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix}, \quad \boldsymbol{v}_3 = \begin{pmatrix} 3 \\ -1 \\ -2 \end{pmatrix}.$$

   (a) Let the linear transformation F be the reflection about the plane spanned by the standard unit vectors $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$. Find the matrix representing F in the standard basis.

   (b) Check that $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\}$ form a basis of $\mathbb{R}^3$.

   (c) Find the matrix representing F in the basis $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\}$.

   (d) Let $\boldsymbol{a} = 4\boldsymbol{v}_1 - 2\boldsymbol{v}_2 + \boldsymbol{v}_3$; compute $F(\boldsymbol{a})$ in the basis $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\}$.

(5+5+9+5)

   (A) Let P_2 be the vector space of polynomials of degree less or equal than 2.

   (a) Show that

$$F(p) = \begin{pmatrix} p(0) \\ p(1) \\ p(2) \end{pmatrix}$$

   is a linear transformation from P_2 into $\mathbb{R}^3$.

   (b) Find the matrix representing F when P_2 is endowed with basis $B = \{1, x, x^2\}$, and $\mathbb{R}^3$ is endowed with its standard basis.

   (c) Find the matrix representing

$$G(p) = \begin{pmatrix} p'(0) \\ p'(1) \\ p'(2) \end{pmatrix}$$

   with respect to the same pair of bases.

   (d) Is F invertible? Is G invertible? Explain.

   (e) Find a matrix A such that

$$A \begin{pmatrix} p(0) \\ p(1) \\ p(2) \end{pmatrix} = \begin{pmatrix} p'(0) \\ p'(1) \\ p'(2) \end{pmatrix}$$

   for every $p \in P_2$.

(5+5+5+5+10)