Engineering and Science Mathematics 2B

Homework 4

due March 3, 2004, before 12:00

Normal questions and advanced questions (A) are worth 5 points; easy questions (E) are worth 4 points. Complete either the easy, or the advanced version, not both.

1. Consider the vector space of functions that is spanned by the basis

$$B = \{\sin x, \cos x, \sin 2x, \cos 2x\} .$$

   Find the matrix representing the derivative operator with respect to the basis B.

2. Recall the definitions of range and kernel of a linear map A on the vector space $\mathbb{R}^n$:

$$\operatorname{Range} A = \{A\boldsymbol{x} \colon \boldsymbol{x} \in \mathbb{R}^n\}$$
$$\operatorname{Ker} A = \{\boldsymbol{x} \in \mathbb{R}^n \colon A\boldsymbol{x} = 0\}$$

   (E) Let

$$A = \begin{pmatrix} -1 & 2 & -2 \\ 1 & 2 & 6 \\ 0 & 1 & 1 \end{pmatrix}$$

   Find a basis for $\operatorname{Range} A$ and for $\operatorname{Ker} A$.

   (A) Prove that $\dim \operatorname{Ker} A + \dim \operatorname{Range} A = n$.

3. Let

$$A = \begin{pmatrix} 2 & 0 & 2 \\ 0 & 2 & 0 \\ -2 & 0 & 2 \end{pmatrix}$$

   represent a linear transformation on $\mathbb{R}^3$ with respect to the standard basis $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3\}$. Find the matrix A' which represents this transformation with respect to the new basis

$$\boldsymbol{e}_1' = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} , \quad \boldsymbol{e}_2' = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} , \quad \boldsymbol{e}_3' = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} .$$

4. Compute the determinant

$$\begin{vmatrix} 1 & 2 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 7 & 3 \\ 0 & 0 & 1 & 1 \end{vmatrix}.$$

5. (E) Use the determinant to test if the matrix

$$A = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 1 & -1 & 1 \end{pmatrix}$$

represents an invertible linear transformation.

(A) Use the definition of the determinant to show that a matrix A is invertible if and only if and only if $\det A \neq 0$.

6. Show that a matrix A is invertible if and only if all the eigenvalues of A are nonzero. (Recall that λ is an eigenvalue of A and $\boldsymbol{v}$ is the corresponding eigenvector if $A\boldsymbol{v} = \lambda\boldsymbol{v}$.)