Engineering and Science Mathematics 2B

Homework 7

due March 31, 2004, before 12:00

Normal questions and advanced questions (A) are worth 5 points; easy questions (E) are worth 4 points. Complete either the easy, or the advanced version, not both.

1. Let $f(t) = \sin t$ on the interval $[0, \frac{\pi}{2})$, and periodically extended outside of this interval, be represented by the complex Fourier series

$$\sum_{n=-\infty}^{\infty} c_n \frac{e^{4nti}}{\sqrt{\frac{\pi}{2}}} \, .$$

   Compute the Fourier coefficients c_n.

2. Find the Fourier transform of $f(x) = e^{-|x|}$.

3. Show that $\mathcal{F}(f(x+a)) = e^{ia\xi} \, \mathcal{F}(f)$.

4. (E) Show that $\mathcal{F}(f') = i\xi \, \mathcal{F}(f)$.
   Hint: Integration by parts. You may assume that all boundary terms are zero when integrating by parts.

   (A) By taking the Fourier transform of the equation

$$\frac{d^2u}{dx^2} - u = f \, ,$$

   show that the solution $u(x)$ can be written as

$$u(x) = \frac{-1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{e^{i\xi x} \, \tilde{f}(\xi)}{1+\xi^2} \, d\xi \, ,$$

   where $\tilde{f}(\xi)$ is the Fourier transform of $f(x)$.

5. Compute the integral

   (E) $\displaystyle\int_{-2}^{2} \delta(2x) \, \cos x \, dx$,

(A) $\displaystyle\int_{-2\pi}^{2\pi} \delta(x^2 - \pi^2)\, \cos x \, dx.$

6. (E) Let the *Heavyside* or *unit step function* be defined by

$$H(x) = \begin{cases} 0 & \text{for } x < 0 \\ \frac{1}{2} & \text{for } x = 0 \\ 1 & \text{for } x > 0\,. \end{cases}$$

Show that $H'(x) = \delta(x)$.
Hint: Integration by parts.

(A) One can define (so-called distributional) derivatives of the δ-function via

$$\int_{-\infty}^{\infty} f(x)\, \delta^{(n)}(x)\, dx = (-1)^n\, f^{(n)}(0)$$

for any n times differentiable function $f(x)$.
So we may be tempted to write a Taylor series for the δ-function,

$$\delta(x + a) = \sum_{n=0}^{\infty} \frac{\delta^{(n)}(x)}{n!}\, a^n\,.$$

It appears that the left side is zero except at $x = -a$, while the right side is zero except at $x = 0$. Resolve this paradox.