Numerical Methods II

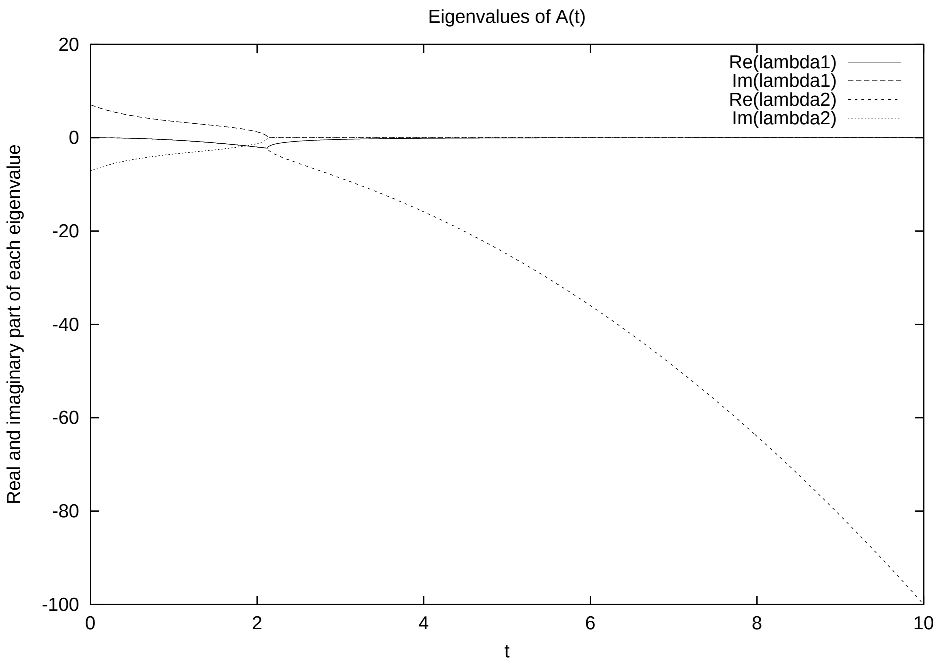
Midterm Exam

March 31, 2004

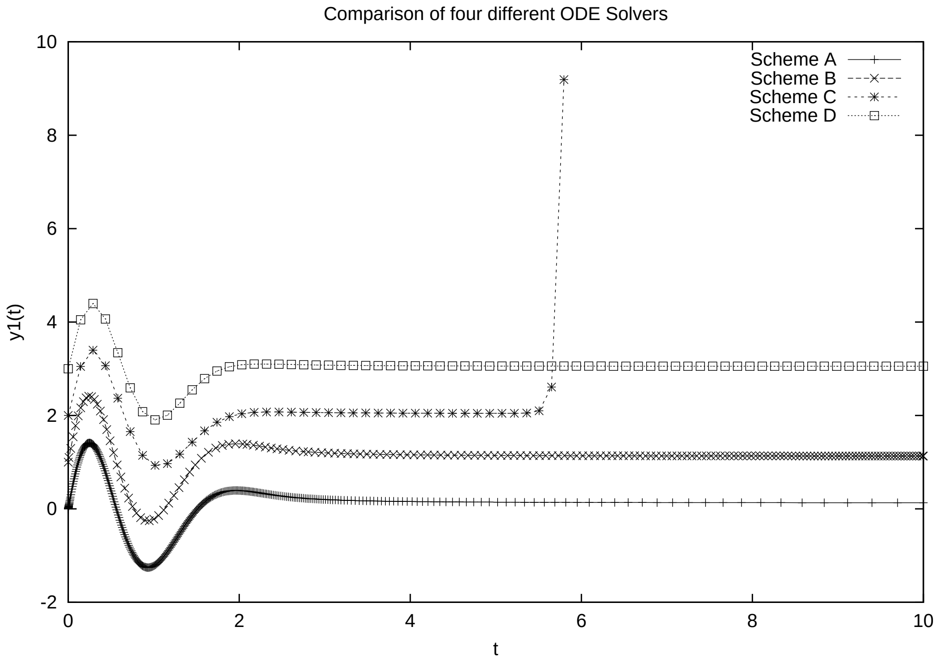
1. You solve the differential equation

$$y'(t) = A(t)\, y(t)\,.$$

(a) Your choice of $A(t)$ is a time-dependent 2×2 matrix with eigenvalues as shown below. For which values of t do you consider the system "stiff"? Explain *briefly*!



(b) You try different solvers from your solver collection: RK4 and BDF4 with constant stepsize, and RK54 and Nordsieck-BDF4 with adaptive stepsize. Match the solvers to the solution graphs below, stating *briefly* one characteristic feature for each.



(Note that the curves are vertically shifted for better legibility, and only one of the components is shown.)

(5+10)

2. For

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} ,$$

find a plane rotation matrix R such that $R^T AR$ is diagonal. (10)

3. Let A be a real symmetric $n \times n$ matrix, and set

$$R(\boldsymbol{x}) = \frac{\boldsymbol{x}^T A \boldsymbol{x}}{\boldsymbol{x}^T \boldsymbol{x}} .$$

(a) If $\boldsymbol{x}$ is an eigenvector of A, show that $R(\boldsymbol{x})$ is the corresponding eigenvalue.

(b) From now on, let $\boldsymbol{x}$ be an arbitrary unit vector, and write

$$\boldsymbol{x} = \sum_{i=1}^{n} \alpha_i \, \boldsymbol{v}_i ,$$

where $\boldsymbol{v}_i$ are the orthonormal eigenvectors of A with corresponding eigenvalues λ_i.

Show that

$$\lambda_{\min} \leq R(\boldsymbol{x}) \leq \lambda_{\max} ,$$

where $\lambda_{\min}$ and $\lambda_{\max}$ denote the smallest and the largest eigenvalue of A, respectively.

(c) Show that

$$\alpha_j = 1 - \frac{1}{2} \|\boldsymbol{x} - \boldsymbol{v}_j\|^2 .$$

(d) **Extra credit:** Finally conclude that

$$R(\boldsymbol{x}) = \lambda_k + O(\|\boldsymbol{x} - \boldsymbol{v}_k\|^2) .$$

(5+5+5+*10*)

4. (a) Let $Q = Q(t)$ be a time-dependent $n \times n$ matrix that satisfies the differential equation

$$Q' = QS , \qquad Q(0) = I , \qquad (*)$$

where $S(t)$ is a skew-symmetric $n \times n$ matrix, i.e. $S^T = -S$.

Show that $Q(t)$ is orthogonal for every $t \geq 0$.

Hint: You have to check that $QQ^T = I$. Differentiate this relation and use the differential equation (*).

(b) **Extra credit:** Let A be a time dependent matrix that satisfies the so-called isospectral flow equation

$$A' = AS - SA , \qquad A(0) = A_0 ,$$

where S is skew symmetric.

Show that the eigenvalues of A remain unchanged under the evolution.

Hint: Use part (a) to conclude that $A(t) = Q^T(t)\, A_0\, Q(t)$.

(10+*10*)

5. You solve the boundary value problem

$$-y''(x) = g(x)$$

on a non-uniform grid. I.e., the step size changes from one node to the next, and we define

$$h_k^+ = x_{k+1} - x_k ,$$
$$h_k^- = x_k - x_{k-1} .$$

(a) Show that the local truncation error for the method

$$-2\left(\frac{y_{k-1}}{h_k^- \, (h_k^- + h_k^+)} - \frac{y_k}{h_k^- \, h_k^+} + \frac{y_{k+1}}{h_k^+ \, (h_k^- + h_k^+)} \right) = g_k \qquad (*)$$

is given by

$$T_k = \tfrac{1}{3}\, y'''(x_k)\, (h_k^- - h_k^+) + O(h_k^-)^2 + O(h_k^+)^2 .$$

(b) What is the (local) order of the method?

(c) **Extra credit:** Replace the right side of (*) by

$$\alpha\, g_{k-1} + (1-\alpha)\, g_{k+1}\,.$$

Determine α so that the local order of the method is at least 2.

(15+5+*10*)