Numerical Methods II

Problem Sets 1/2

due in class, February 23, 2004

1. (From Gautschi, 1997, p. 324.)

   (a) Show that any one-step method of order p, when applied to the linear model problem $\boldsymbol{y}' = A\boldsymbol{y}$, yields
   $$\boldsymbol{y}_{n+1} = \phi(hA)\,\boldsymbol{y}_n\,,$$
   where
   $$\phi(z) = 1 + z + \tfrac{1}{2!}\,z^2 + \cdots + \tfrac{1}{p!}\,z^p + O(z^{p+1})\,.$$

   (b) Show, in particular, that the $O(z^{p+1})$ terms vanish for a p-stage Runge–Kutta method of order $p = 1, \ldots, 4$, and for the Taylor method of order $p \geq 1$.

2. (From Gautschi, 1997, p. 328.) For an analytic function f, show that the level curve $|f(z)| = 1$ obeys the differential equation
   $$\frac{\mathrm{d}z}{\mathrm{d}s} = \mathrm{i}\,\frac{f(z)\,|f'(z)|}{f'(z)}\,,$$
   where s is the arclength parameter along the level curve.

   Hint: Write $f(z) = r\,\mathrm{e}^{\mathrm{i}\theta}$ and take θ as the independent variable. In a second step, change the independent variable from θ to s.

3. **Project:** Use the results from the previous two exercises to plot the boundary of the region of absolute stability in the complex $z = \lambda h$ plane for the Taylor methods of order $p = 1, \ldots, 5$.

   Notes: You can use, for example, the integrator `ode_rk4` from last semester. The origin $z = 0$ is always on the boundary of the region of absolute stability. (Explain!)

4. *Shift of the Taylor polynomial.*

   (a) Let $p(t)$ be a polynomial of degree m, and let $\boldsymbol{v}$ be the vector of coefficients of its Taylor expansion about $t = 0$, scaled such that
   $$p(t) = v_0 + v_1\,\frac{t}{h} + \cdots + v_m\left(\frac{t}{h}\right)^m .$$

Show that the scaled Taylor coefficients of p about the point $t = h$, i.e. the coefficients defined via

$$p(t) = w_0 + w_1 \, \frac{t-h}{h} + \cdots + w_m \left(\frac{t-h}{h} \right)^m ,$$

are given by $\boldsymbol{w} = S\boldsymbol{v}$, where the components of S are

$$S_{ij} = \begin{cases} 0 & \text{for } i > j \\ \binom{j}{i} & \text{for } i \leq j \end{cases}$$

and $i, j = 0, \ldots, m$.

(b) Show that the inverse shift has coefficients

$$(S^{-1})_{ij} = \begin{cases} 0 & \text{for } i > j \\ (-1)^{j-i} \binom{j}{i} & \text{for } i \leq j \, . \end{cases}$$

5. *BDF method in Nordsieck's representation.* Recall that BDF methods are implicit linear multistep methods obtained through approximating the left side of the equation $y' = f(t, y)$ by a polynomial interpolant.

More precisely, for the m-step, order m BDF method in the scalar case, let p_n denote the interpolating polynomial at timestep t_n. Then

$$p_n(t_{n-j}) = y_{n-j} \quad \text{for } j = 0, \ldots, m \, ,$$
$$p_n'(t_n) = f_n \equiv f(t_n, y_n) \, .$$

Thus, computing the numerical solution y_n at time $t = t_n$ requires knowledge of m previous values $y_{n-m}, \ldots, y_{n-1}$.

Nordsieck's idea is that instead of storing previous values, one may as well store the scaled Taylor coefficients of the interpolating polynomial, namely

$$\boldsymbol{z}_n = \begin{pmatrix} p_n(t_n) \\ p_n'(t_n) \, h \\ \vdots \\ p_n^{(m)}(t_n) \, h^m/m! \end{pmatrix} .$$

(a) Show that

$$p_{n+1}(t) - p_n(t) = (y_{n+1} - p_n(t_{n+1})) \, \phi\Big(\frac{t - t_{n+1}}{h} \Big) ,$$

where

$$\phi(z) = \prod_{j=1}^m \Big(\frac{z}{j} + 1 \Big) .$$

(b) Show that the BDF method of order m is equivalent to

$$\boldsymbol{z}_{n+1} = S\boldsymbol{z}_n + \boldsymbol{v}\left(h\, f_{n+1} - \boldsymbol{e}_1^T S\boldsymbol{z}_n\right),$$

where S is the shift matrix from Question 4, $\boldsymbol{v}$ is the vector of (unscaled) Taylor coefficients of the polynomial $\phi(z)/\phi'(0)$, and $\boldsymbol{e}_1 = (0, 1, 0, \ldots, 0)^T$.

(c) Describe how you would change the step size in Nordsieck's representation of the BDF method.

6. **Project:** Implement BDF4 with step size control for a system of linear equations in Nordsieck's representation as an Octave function. The solver should take arguments of the form

```
ode_bdf4_var ('f', [t0,t1], y0, tol, Nmax)
```

where `f` is the function $f(t, y)$ on the right hand side of the differential equation, `t0` is the initial time, `t1` is the final time, `y0` the initial condition, `tol` the local error tolerance, and `Nmax` the maximum number of steps.

Estimate the local error by using a predictor-corrector pair, where the predictor is the extrapolated value $p_n(t_{n+1})$, and the corrector is given by the result of the BDF step. Use Broyden's method to solve the system of nonlinear equations that arises at each time step.

7. **Project:** Use your variable time-step BDF method to solve the Van der Pol equation

$$\begin{aligned}\dot{x} &= y\,,\\ \dot{y} &= \mu\,(1 - x^2)\,y - x\,,\end{aligned}$$

with initial data $x(0) = 2$ and $y(0) = 0$. Can you get up to $\mu = 1000$ within a reasonable computation time?