Numerical Methods II

Problem Set 4

due in class, March 10, 2004

(From SM, Chapter 5)

1. Show that if H_k is a $k \times k$ Householder matrix, i.e.

   $$H_k = I - 2\,\frac{\boldsymbol{v}_k \boldsymbol{v}_k^T}{\boldsymbol{v}_k^T \boldsymbol{v}_k}\,,$$

   then

   $$H = \begin{pmatrix} I_{n-k} & 0 \\ 0 & H_k \end{pmatrix}$$

   is an $n \times n$ Householder matrix.

2. Use Householder matrices to transform the matrix

   $$A = \begin{pmatrix} 2 & 1 & 2 & 2 \\ 1 & -7 & 6 & 5 \\ 2 & 6 & 2 & -5 \\ 2 & 5 & -5 & 1 \end{pmatrix}$$

   into tridiagonal form.

3. Let $A^{(0)}$ be a diagonal $n \times n$ matrix with distinct diagonal elements, and $A^{(1)}$ be a symmetric $n \times n$ matrix. Assume that

   $$A_\varepsilon \equiv A^{(0)} + \varepsilon\, A^{(1)}$$

   has an eigenvalue λ_ε with corresponding eigenvector $\boldsymbol{x}_\varepsilon$. Show that

   $$\lambda_\varepsilon = \lambda^{(0)} + \varepsilon\,\lambda^{(1)} + O(\varepsilon^2)\,,$$
   $$\boldsymbol{x}_\varepsilon = \boldsymbol{x}^{(0)} + \varepsilon\,\boldsymbol{x}^{(1)} + O(\varepsilon^2)\,,$$

   where $\lambda^{(0)} = a_{jj}^{(0)}$ for some $j = 1, \ldots, n$, $\lambda^{(1)} = a_{jj}^{(1)}$, $\boldsymbol{x}^{(0)} = \boldsymbol{e}_j$, and

   $$x_i^{(1)} = \frac{a_{ij}^{(1)}}{a_{jj}^{(0)} - a_{ii}^{(0)}}$$

   for $i \neq j$. Explain why normalizability of the eigenvector requires that we choose $x_j^{(1)} = 0$.

4. *(QR iteration.)* Let A be a symmetric, tridiagonal $n \times n$ matrix, and assume that $A - \mu I = QR$ where Q is a product of plane rotations and R is upper triangular and also tridiagonal. Show that

   (a) $B = RQ + \mu I$ is symmetric;

   (b) B has the same eigenvalues as A;

   (c) B is tridiagonal.