Numerical Methods II

Problem Set 5

due in class, March 24, 2004

1. (From SM, p. 177.) Perform one step of the QR algorithm, using the shift $\mu = a_{nn}$, for the matrix
$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} .$$
Show that the QR algorithm does not converge for this matrix. (A different shift must be used for this matrix.)

2. (From SM, p. 177.) An eigenvalue and eigenvector of the $n \times n$ matrix A may be evaluated by solving the system of nonlinear equations
$$\begin{aligned} (A - \lambda I)\, \boldsymbol{x} &= 0\,, \\ \boldsymbol{x}^T \boldsymbol{x} &= 1 \end{aligned}$$
for the $n+1$ unknowns λ and $\boldsymbol{x}$. Show that, when applying Newton's method to this problem, you obtain the iteration
$$\begin{aligned} (A - \lambda_k I)\, \Delta\boldsymbol{x}_k - \Delta\lambda_k\, \boldsymbol{x}_k &= -(A - \lambda_k I)\, \boldsymbol{x}_k\,, \\ -\boldsymbol{x}_k^T \Delta\boldsymbol{x}_k &= \tfrac{1}{2}\, (\boldsymbol{x}_k^T \boldsymbol{x}_k - 1)\,, \end{aligned}$$
where
$$\begin{aligned} \boldsymbol{x}_{k+1} &= \boldsymbol{x}_k + \Delta\boldsymbol{x}_k\,, \\ \lambda_{k+1} &= \lambda_k + \Delta\lambda_k\,. \end{aligned}$$
Comment on the difference between this method and the method of inverse iteration.

3. **Project:** Write a simple Octave function that performs the QR algorithm with shift. You may use the built-in function `qr` to perform the QR decomposition. Test your code on the matrices
$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \text{and} \qquad A = \begin{pmatrix} 5 & 1 \\ 1 & 7 \end{pmatrix} .$$
Experiment with the shifts and describe how different shifting strategies affect the rate of convergence.

4. **Project:** The method of inverse iteration can approximate eigenvectors even when the corresponding eigenvalue is not accurately known. Describe a method for finding the eigenvalue once the eigenvector is computed, and test your method with the matrices from the previous question.

5. Derive the update rule in Brent's method for finding the minimum of a scalar function: Given three evaluation points x_0, x_1, and x_2 with corresponding function values $y_0 = f(x_0)$, $y_1 = f(x_1)$, and $y_2 = f(x_2)$ (usually taken from the bracket about the minimum), find the quadratic polynomial that interpolates between these points. Hence, find the expression for the extremum of this polynomial, to be taken as the new evaluation point x_n.