HW6 – Solutions

1a. Set $x = \cos\theta$

$$\Rightarrow T_k(x) = \cos(k\theta) = \operatorname{Re} e^{ik\theta} = \tfrac{1}{2} e^{ik\theta} + \tfrac{1}{2} e^{-ik\theta}$$

Now, $$e^{i\theta} = \cos\theta + i\sin\theta = x + i\sqrt{1-x^2}$$

$$\Rightarrow T_k(x) = \tfrac{1}{2}\left(x + i\sqrt{1-x^2}\right)^k + \tfrac{1}{2}\left(x - i\sqrt{1-x^2}\right)^k$$

Note: Strictly speaking, the computation is valid only for $x \in [-1,1]$. However, expanding the result shows that the expression is a polynomial (which is not obvious at first sight) and thus extends to all $x \in \mathbb{R}$.

(b) Defining the spectral condition number $\mathcal{K} = \lambda_{max}/\lambda_{min}$, we see that

$$\frac{\lambda_{max} + \lambda_{min}}{\lambda_{max} - \lambda_{min}} = \frac{\mathcal{K}+1}{\mathcal{K}-1} =: x$$

Therefore,

$$x \pm i\sqrt{1-x^2} = x \pm \sqrt{x^2-1} = \frac{\mathcal{K}+1}{\mathcal{K}-1} \pm \sqrt{\frac{(\mathcal{K}+1)^2}{(\mathcal{K}-1)^2} - \frac{(\mathcal{K}-1)^2}{(\mathcal{K}-1)^2}}$$

$$= \frac{(\mathcal{K}+1) \pm \sqrt{4\mathcal{K}}}{\mathcal{K}-1} = \frac{(\sqrt{\mathcal{K}} \pm 1)^2}{(\sqrt{\mathcal{K}}+1)(\sqrt{\mathcal{K}}-1)} = \frac{\sqrt{\mathcal{K}} \pm 1}{\sqrt{\mathcal{K}} \mp 1}$$

$$\Rightarrow T_k(x) = \frac{1}{2}\left(\frac{\sqrt{\mathcal{K}}+1}{\sqrt{\mathcal{K}}-1}\right)^k + \frac{1}{2}\left(\frac{\sqrt{\mathcal{K}}-1}{\sqrt{\mathcal{K}}+1}\right)^k$$

$$\geq \frac{1}{2}\left(\frac{\sqrt{\mathcal{K}}+1}{\sqrt{\mathcal{K}}-1}\right)^k$$

$$\Rightarrow \boxed{\frac{1}{T_k(x)} \leq 2\left(\frac{\sqrt{\mathcal{K}}-1}{\sqrt{\mathcal{K}}+1}\right)^k}$$

2a) Let $v_1, \dots, v_n$ be an orthonormal basis of eigenvectors of A.

We can thus write the true solution of $Ax = b$ in the form

$$x = \sum_{j=1}^{n} \alpha_j v_j$$

$$\Rightarrow b = Ax = \sum_{j=1}^{n} \lambda_j \alpha_j v_j$$

Let $x^i = \sum_{j=1}^{n} \alpha_j^i v_j$ denote the i-th iterate of the algorithm, where $\{\alpha_j^1\}$ are chosen arbitrarily.

We compute

$$\begin{aligned} r^i &= b - Ax^i \\ &= \sum_{j=1}^{n} \lambda_j \alpha_j v_j - \sum_{j=1}^{n} \lambda_j \alpha_j^i v_j \\ &= \sum_{j=1}^{n} \lambda_j (\alpha_j - \alpha_j^i) v_j \end{aligned}$$

At the i-th step, we pick an eigenvalue λ_i, and compute

$$\begin{aligned} x^{i+1} &= x^i + \frac{r^i}{\lambda_i} \\ &= \sum_{j=1}^{n} \alpha_j^i v_j + \sum_{j=1}^{n} (\alpha_j - \alpha_j^i) \frac{\lambda_j}{\lambda_i} v_j \end{aligned}$$

We see that

$$\alpha_j^{i+1} = \alpha_j \quad \text{if } \lambda_i = \lambda_j \quad \text{or if } \alpha_j^i = \alpha_j$$

$\Rightarrow$ After all d distinct eigenvalues have been encountered, we have $\alpha_j^{i+1} = \alpha_j$ for all $j = 1, \dots, n$.

(b) To get close to the solution after less than d iterations, one should use the eigenvalues in the order that assures that the error is decreased in all the eigenspaces, i.e. $\frac{\lambda_j}{\lambda_i}$ should be less than one for those eigenspaces for which $\alpha_j^i \neq \alpha_j$.

$\Rightarrow$ Take the eigenvalues from large to small.

(c) If rounding error results in $\alpha_j^i \neq \alpha_j$ although the corresponding eigenvalue λ_j has already been used, then this error will be amplified by a factor $\frac{\lambda_j}{\lambda_i}$ in each subsequent iteration.

(d) To minimize rounding errors, we have to insure that $\frac{\lambda_j}{\lambda_i} < 1$ for eigenspaces for which $\alpha_j^i \approx \alpha_j$.

$\Rightarrow$ Take eigenvalues from small to large.

3. $$\beta_k = - \frac{r_k^T A d_{k-1}}{d_{k-1}^T A d_{k-1}}$$

Recall that

$$r_k = r_{k-1} - \alpha_{k-1} A d_{k-1}, \quad \text{where } \alpha_{k-1} = \frac{d_{k-1}^T r_{k-1}}{d_{k-1}^T A d_{k-1}}$$

$$\Rightarrow A d_{k-1} = \frac{r_{k-1} - r_k}{\alpha_{k-1}} = (r_{k-1} - r_k) \frac{d_{k-1}^T A d_{k-1}}{d_{k-1}^T r_{k-1}}$$

$$\Rightarrow \beta_k = - \frac{r_k^T (r_{k-1} - r_k) \dfrac{d_{k-1}^T A d_{k-1}}{d_{k-1}^T r_{k-1}}}{d_{k-1}^T A d_{k-1}}$$

$$= \frac{r_k^T (r_k - r_{k-1})}{d_{k-1}^T r_{k-1}}$$

Since $d_{k-1} = r_{k-1} + \beta_{k-1} \underbrace{d_{k-2}}_{\in V_{k-1}}$,

and $r_{k-1}^T v = 0$ for all $v \in V_{k-1}$, we see that

$$d_{k-1}^T r_{k-1} = r_{k-1}^T r_{k-1},$$

and therefore

$$\beta_k = \frac{r_k^T (r_k - r_{k-1})}{r_{k-1}^T r_{k-1}} \equiv \beta_k^{PR}.$$

Similarly $r_{k-1} \in V_k$ and $r_k^T v = 0$ for all $v \in V_k$,

so that

$$\beta_k = \frac{r_k^T r_k}{r_{k-1}^T r_{k-1}} \equiv \beta_k^{FR}$$