Midterm 2 Solutions.

1. The semi-discrete Fourier transform of a function $f \in \ell^1(\mathbb{Z})$ is defined

$$\hat{f}(k) = \sum_{j\in\mathbb{Z}} f[j]\, e^{-2\pi i j k}.$$

Now apply $\sum_{m=0}^{N-1} e^{2\pi i \frac{nm}{N}}$ to both sides of the equation with $k = \frac{m}{N}$

$$\sum_{m=0}^{N-1} \hat{f}\left(\tfrac{m}{N}\right) e^{2\pi i \frac{nm}{N}} = \sum_{m=0}^{N-1} e^{2\pi i \frac{nm}{N}} \sum_{j\in\mathbb{Z}} f[j]\, e^{-2\pi i j \frac{m}{N}}$$

$$= \sum_{j\in\mathbb{Z}} f[j] \underbrace{\sum_{m=0}^{N-1} e^{2\pi i \frac{n-j}{N} m}}_{= \begin{cases} N & \text{if } n-j = 0 \bmod N \\ 0 & \text{if } n-j \neq 0 \bmod N \end{cases}}$$

(as shown in class)

$$= N \sum_{r\in\mathbb{Z}} f[n + rN]$$

(This is yet another version of the Poisson summation formula)

Interpretation: For a non-periodic function on an (infinite) grid, the Fourier transform is a periodic, continuous function. If we periodicize the function – this is what's done on the LHS of the expression on the exam sheet – then we need to retain information of the Fourier transform only on a finite, discrete set.

2. a) We seek solutions u_0 s.t. $\partial_t u_0 = \partial_x u_0 = \Delta u_0 = 0$, i.e.

$$u_0 - u_0^2 = 0 \qquad (*)$$

$\Rightarrow$ $u_0 = 0$ or $u_0 = 1$.

b) We linearize the equation as follows. With $u = u_0 + \delta\tilde{u}$,

$$\delta \partial_t \tilde{u} = \delta\varepsilon\Delta\tilde{u} + (u_0 + \delta\tilde{u})(1 - i\delta\partial_x\tilde{u}) - (u_0 + \delta\tilde{u})^2$$

$$= \delta\varepsilon\Delta\tilde{u} + u_0 + \delta\tilde{u} - i\delta u_0 \partial_x \tilde{u} - u_0^2 - 2u_0\delta\tilde{u} + O(\delta$$

Dropping terms of order δ^2 and using $(*)$, we find

$$\partial_t\tilde{u} = \varepsilon\Delta\tilde{u} + \tilde{u} - iu_0\partial_x\tilde{u} - 2u_0\tilde{u}$$

Now insert the ansatz $\tilde{u} = e^{\lambda t} e^{ikx}$, so that

$$\lambda\tilde{u} = -\varepsilon k^2\tilde{u} + \tilde{u} + u_0 k\tilde{u} - 2u_0\tilde{u}$$

$$\Rightarrow \quad \lambda = -\varepsilon k^2 + u_0 k + 1 - 2u_0$$

For $u_0 = 0$,

$$\lambda = -\varepsilon k^2 + 1$$

$\Rightarrow$ The mode $k=0$ is most unstable

For $u_0 = 1$,

$$\lambda = -\varepsilon k^2 + k - 1$$

The growth rate is maximal when $\frac{d\lambda}{dk} = -2\varepsilon k + 1 = 0$

$$\Rightarrow \quad k = \frac{1}{2\varepsilon}$$

whence $\lambda_{max} = -\frac{\varepsilon}{4\varepsilon^2} + \frac{1}{2\varepsilon} - 1$

$$= \frac{1}{4\varepsilon} - 1$$

Turing instability means that $\lambda_{max} > 0$, i.e. $\frac{1}{4\varepsilon} > 1 \Rightarrow \boxed{\varepsilon < \frac{1}{4}}$.

3. Total mass is $M = \int_a^b u(x,t)\, dx$.

Integrate equation over the domain:

$$\dot{M} + v \underbrace{\int_a^b \partial_x u \, dx}_{= u(b) - u(a)} = D \underbrace{\int_a^b \partial_{xx} u \, dx}_{= \partial_x u(b) - \partial_x u(a)}$$

So, mass is conserved if

$$v(u(b) - u(a)) = D(\partial_x u(b) - \partial_x u(a))$$

If we want to impose local boundary conditions, we could require

$$u = \frac{D}{v} \partial_x u$$

at both end-points.

4. Use characteristics: Set

$$V(t;a) = u(z(t), t)$$

where $z(0) = a$

$$\Rightarrow \partial_t V = \partial_t u(z,t) + \dot{z}\, \partial_x u(z,t) \qquad \text{by the chain rule}$$

If we set $\dot{z} = v$, i.e. $z = a + vt$, then

$$\partial_t V = \partial_t u(z,t) + v\, \partial_x u(z,t) \underset{\uparrow}{=} D\, \partial_{xx} u(z,t) = D\, \partial_{aa}(u(a+vt,t))$$

use advection-diffusion equation

$$\Rightarrow \quad \partial_t V = D\, \partial_{aa} V$$

and we can apply the solution formula to V:

$$V(a,t) = \frac{1}{\sqrt{4\pi D t}} \int_{-\infty}^{\infty} e^{-\frac{(a-b)^2}{4Dt}} \underbrace{V(b,0)}_{= u(b,0) = g(b)} \, db$$

Since

$$V(a,t) = u(a+vt, t),$$

$$u(x,t) = V(x-vt, t)$$

$$\Rightarrow \quad \boxed{u(x,t) = \frac{1}{\sqrt{4\pi D t}} \int_{-\infty}^{\infty} e^{-\frac{(x-vt-b)^2}{4Dt}} g(b)\, db}$$