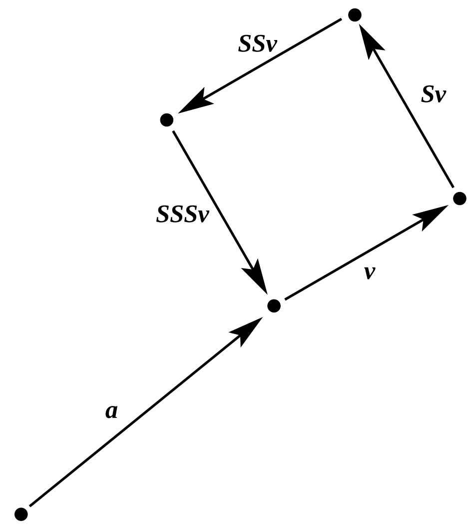
Solution to Ivanov, Chapter 3, Problem 9

Let $S\boldsymbol{v}$ denote the rotation by 90 degrees anticlockwise of the vector $\boldsymbol{v}$. Further, define

$$S_i\boldsymbol{v} = \sum_{j=1}^{i} S^{j-1}\boldsymbol{v} \tag{1}$$

where $S_0\boldsymbol{v} = \boldsymbol{0}$. Then for any vector $\boldsymbol{v}$, the set $\{S_0\boldsymbol{v}, S_1\boldsymbol{v}, S_2\boldsymbol{v}, S_3\boldsymbol{v}\}$ are the four vertices of a square. Vice versa, any square can be represented of a translation of this set for some vector $\boldsymbol{v}$; see the figure below.



It is easy to check that $S_i(\boldsymbol{v}+\boldsymbol{w}) = S_i\boldsymbol{v} + S_i\boldsymbol{w}$ for any two vectors $\boldsymbol{v}$ and $\boldsymbol{w}$. (In fact, S_i is a linear map and can be though of as a matrix.)

The solution to Problem 9 is now an easy application of vector algebra. Let the vertices of the first square have coordinates

$$\boldsymbol{p}_i = \boldsymbol{a} + S_i\boldsymbol{v} \quad \text{for } i = 0, \dots, 3\,, \tag{2}$$

and the vertices of the second square have coordinates

$$\boldsymbol{q}_i = \boldsymbol{b} + S_i\boldsymbol{w} \quad \text{for } i = 0, \dots, 3\,. \tag{3}$$

Then the midpoint of the line from the ith vertex of the first square to the ith vertex of the second square (the above formulas make no assumption on where we start counting, but once a first vertex is specified, follow anti-clockwise orientation) has coordinates

$$\frac{\boldsymbol{p}_i + \boldsymbol{q}_i}{2} = \frac{\boldsymbol{a} + S_i\boldsymbol{v} + \boldsymbol{b} + S_i\boldsymbol{w}}{2} = \frac{\boldsymbol{a}+\boldsymbol{b}}{2} + S_i\left(\frac{\boldsymbol{v}+\boldsymbol{w}}{2}\right). \tag{4}$$

This is again the representation for the vertices of a square. $\square$