General Mathematics and Computational Science II

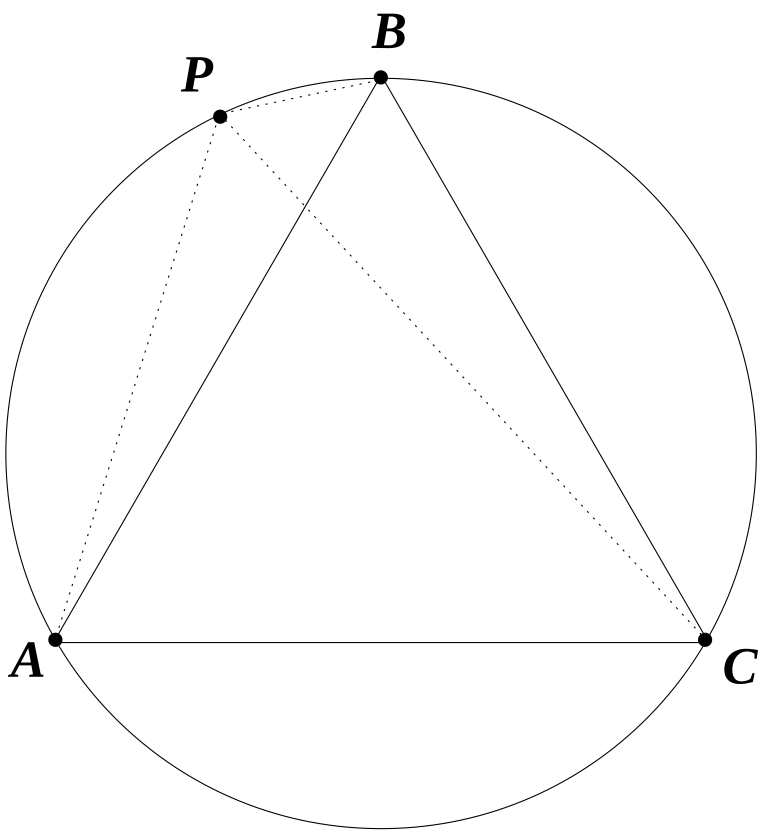
Midterm Exam

March 6, 2007

1. A point P lies on the arc AB of the circle circumscribing an equilateral triangle ABC; see figure. Prove that $PC = PA + PB$.

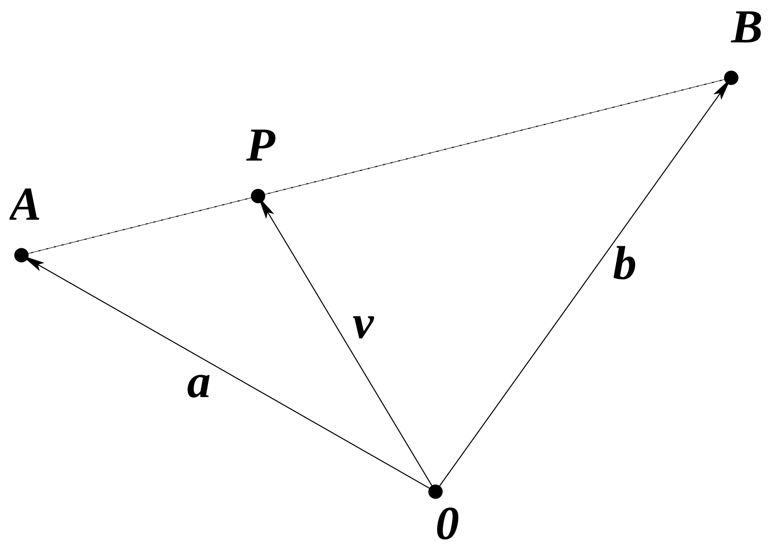
   Hint: By the *arc central angle theorem*, the angles $\angle APC$ and $\angle ABC$ are equal.

   (10)



2. A point P lies on the line segment AB such that $AP = 2\,PB$; see figure. The coordinates of the points are given by $\boldsymbol{a}$, $\boldsymbol{v}$, and $\boldsymbol{b}$, respectively.

   Find an expression for $\boldsymbol{v}$ in terms of $\boldsymbol{a}$ and $\boldsymbol{b}$. (5)



3. Describe the symmetry group of the square. How many elements does it have? Give an example of a minimal generating set. (10)

4. (a) Are the even integers $\{2k \colon k \in \mathbb{Z}\}$ a subgroup of $(\mathbb{Z}, +)$? The odd integers $\{2k+1 \colon k \in \mathbb{Z}\}$?

   (b) Show that the identity element e of a group $(G, \circ)$ is the unique element which satisfies $e \circ a = a \circ e = a$ for all $a \in G$.

(5+5)

5. Consider an ensemble of Kac rings with N sites. Initially, each member of the ensemble has the same configuration of black and white balls, while the distribution of markers among the edges of the rings is random; each edge carries a marker with probability μ. Recall from class that the difference between the number of black balls and the number of white balls, averaged over the ensemble, is given by

$$\langle \Delta(t) \rangle = (1 - 2\,\mu)^t \, \Delta(0)$$

for $0 \leq t \leq N$.

   (a) Show that $\mathrm{Var}[\Delta(N)] = (1 - (1-2\mu)^{2N})\, \Delta^2(0)$.

   (b) Is $\langle \Delta(N) \rangle$ a good description of $\Delta(N)$ for a "typical" member of the ensemble? Briefly explain your answer.

(7+3)

6. Set up a Kac ring in its initial state S_0 with a random distribution of markers and balls and assume that Δ, the difference between the number of black and the number of white balls, is positive.

First, turn the ring by one clockwise step into some state S_1. We demonstrated in class that Δ is more likely to decrease than to increase during this first step.

Now turn the ring one step anti-clockwise. Since the dynamics is time-reversible, we are reverting back to S_0, hence Δ is more likely to increase than to decrease in this backward step.

Now turn the ring another step anti-clockwise into a state S_{-1}. Is Δ now more likely to increase or to decrease? Explain! (5)