General Mathematics and Computational Science II

Midterm Exam

March 6, 2007

Last Name: ______________________________

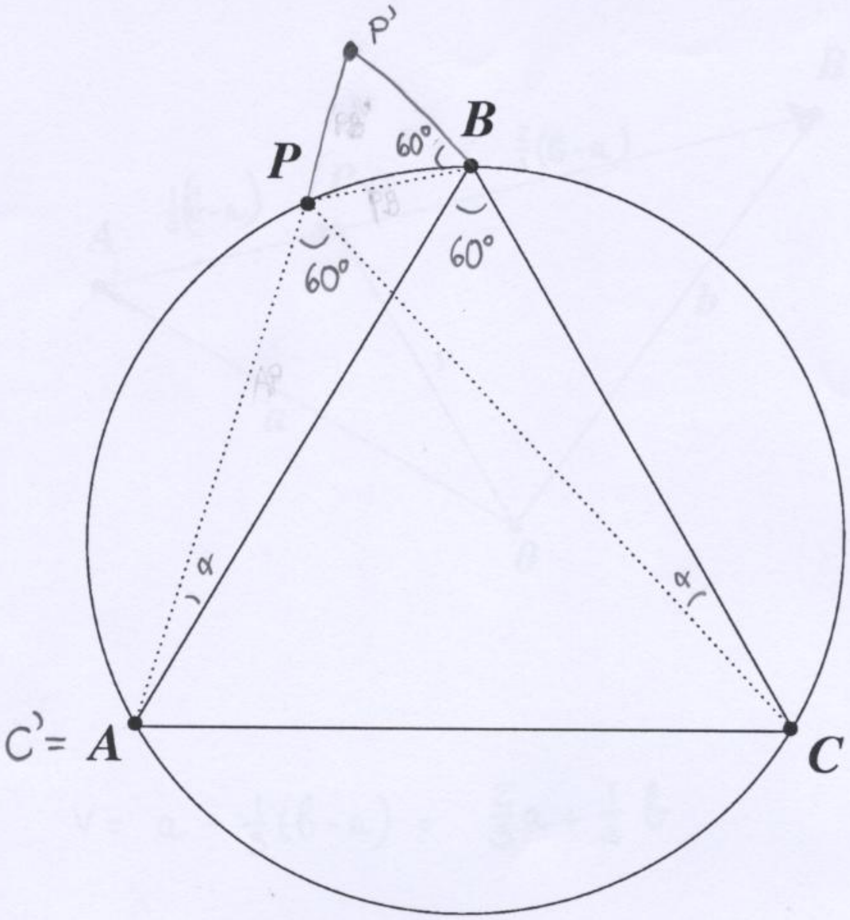
First Name: ______________________________

Signature: ______________________________

1. A point P lies on the arc AB of the circle circumscribing an equilateral triangle ABC; see figure. Prove that PC = PA + PB.

Hint: By the *arc central angle theorem*, the angles $\angle APC$ and $\angle ABC$ are equal.

(10)



The hint implies that $\measuredangle PAB = \measuredangle PCB = \alpha$.

Thus, if we rotate by 60° clockwise about B, the image of PC goes through P, so that

$$PC = AP' = AP + PP'$$

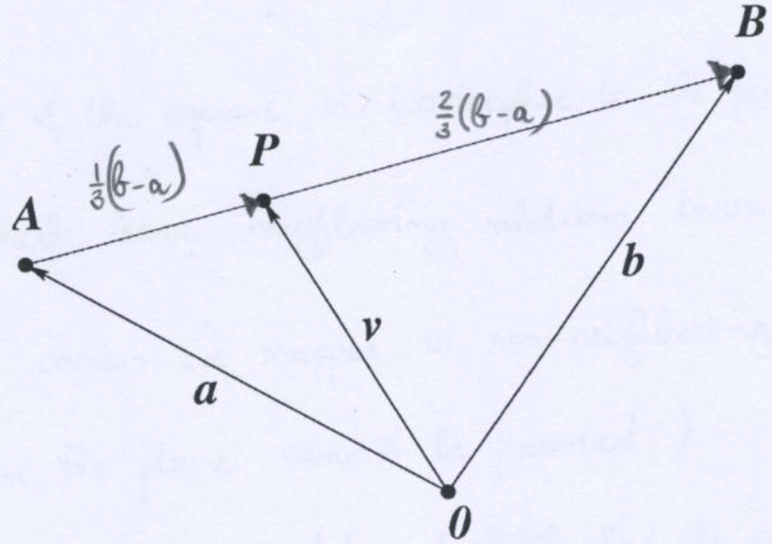
Clearly, the triangle PP'B is equilateral, i.e. $PP' = PB$

$$\Rightarrow \quad PC = AP + PB$$

□

2. A point P lies on the line segment AB such that $AP = 2\,PB$; see figure. The coordinates of the points are given by $\mathfrak{a}$, $\mathfrak{v}$, and $\mathfrak{b}$, respectively.

   Find an expression for $\mathfrak{v}$ in terms of $\mathfrak{a}$ and $\mathfrak{b}$. (5)



From picture: $V = a + \frac{1}{3}(b-a) = \frac{2}{3}a + \frac{1}{3}b$

3. Describe the symmetry group of the square. How many elements does it have? Give an example of a minimal generating set. (10)

The symmetry group of the square is isomorphic to the permutations of its corners which leave neighboring relations invariant.

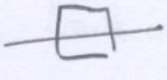
(If neighboring corners are mapped to non-neighboring ones, their distance in the plane cannot be preserved.)

These are 4 cyclical shifts (rotations by $k \cdot 90°$ about the center of the square) and order reversal (reflections about an axis of symmetry), so there are $4 \cdot 2 = 8$ elements altogether.

A minimal set of generators is

{rotation by $90°$ about the center, reflection about one diagonal}

or, alternatively

{reflection about , reflection about }.

4. (a) Are the even integers $\{2k\colon k \in \mathbb{Z}\}$ a subgroup of $(\mathbb{Z}, +)$? The odd integers $\{2k+1\colon k \in \mathbb{Z}\}$?

(b) Show that the identity element e of a group $(G, \circ)$ is the unique element which satisfies $e \circ a = a \circ e = a$ for all $a \in G$.

(5+5)

(a) The even integers are a subgroup:

- the sum of two even numbers is even
- the inverse of an even number $2k$, which is $-2k$, is even

The odd integers are not a subgroup (clearly not closed under addition and no identity element)

(b) Assume there is another $f \in G$ s.t. $f \circ a = a \circ f = a \quad \forall a \in G$ $(*)$

Since $f \in G$, it must have an inverse, so

$$e = f \circ f^{-1} \qquad (**)$$

with $(*)$, we conclude $e = f^{-1}$, substitution back into $(**)$ gives

$$e = f \circ e = f$$

↑ e is identity

$\square$

5. Consider an ensemble of Kac rings with N sites. Initially, each member of the ensemble has the same configuration of black and white balls, while the distribution of markers among the edges of the rings is random; each edge carries a marker with probability μ. Recall from class that the difference between the number of black balls and the number of white balls, averaged over the ensemble, is given by

$$\langle \Delta(t) \rangle = (1 - 2\mu)^t \Delta(0)$$

for $0 \leq t \leq N$.

(a) Show that $\mathrm{Var}[\Delta(N)] = (1 - (1 - 2\mu)^{2N})\,\Delta^2(0)$.

(b) Is $\langle \Delta(N) \rangle$ a good description of $\Delta(N)$ for a "typical" member of the ensemble? Briefly explain your answer.

(7+3)

(a) After N steps, each ball has passed every marker exactly once.

Thus, $\Delta(N) = \pm\,\Delta(0)$

(the sign depending on whether the number of markers is even or odd), so that

$$\Delta^2(N) = \Delta^2(0) = \langle \Delta^2(0) \rangle$$

$$\Rightarrow \mathrm{Var}\left[\Delta(N)\right] = \langle \Delta^2(N) \rangle - \langle \Delta(N) \rangle^2$$

$$= \langle \Delta^2(0) \rangle - (1-2\mu)^{2N}\,\Delta^2(0)$$

$$= \left(1 - (1-2\mu)^{2N}\right)\Delta^2(0)$$

(b) No, the standard deviation (root of the variance) is almost as large as the initial Δ, this is the situation where individual and averaged behavior are almost maximally different.

(See graphs in handout)

6. Set up a Kac ring in its initial state S_0 with a random distribution of markers and balls and assume that Δ, the difference between the number of black and the number of white balls, is positive.

   First, turn the ring by one clockwise step into some state S_1. We demonstrated in class that Δ is more likely to decrease than to increase during this first step.

   Now turn the ring one step anti-clockwise. Since the dynamics is time-reversible, we are reverting back to S_0, hence Δ is more likely to increase than to decrease in this backward step.

   Now turn the ring another step anti-clockwise into a state S_{-1}. Is Δ now more likely to increase or to decrease? Explain! (5)

With regard to the random initial state, a clockwise step is no different than an anticlockwise step, so the transition $S_0 \to S_1$ must have the same ensemble properties as the transition $S_0 \to S_{-1}$.

What is odd, at first, is that $S_1 \to S_0$ is behaving qualitatively different than $S_0 \to S_{-1}$. Note, however, that S_1 is NOT a random initial state, but one that has been obtained by a forward step from S_0. As such, it is special among all states with same Δ. Thus, it is no contradiction that Δ is likely to increase during $S_1 \to S_0$.