Partial Differential Equations

Final Exam

May 9, 2007

1. Consider the homogeneous Helmholtz equation on $\mathbb{R}^3$,

$$(1-\Delta)u = 0\,.$$

   (a) Show that if u is a radial solution, i.e. $u(x) = v(r)$ with $r = |x|$, then

$$v - \frac{2}{r}\,v' - v'' = 0\,.$$

   (b) Show that

$$v_\pm(r) = \frac{\mathrm{e}^{\pm r}}{r}$$

   are two independent radial solutions. Which one would you consider to use as a fundamental solution for the Helmholtz equation?

(10+10)

2. Let $U \subset \mathbb{R}^n$ be open and assume that $u\colon U \to \mathbb{R}$ is harmonic.

   (a) Show that, for any $i = 1, \ldots, n$,

$$|\partial_i u(x)| \leq \frac{n}{r}\,\|u\|_{L^\infty(\partial B(x,r))}$$

   so long as $B(x,r) \subset U$.

   (b) Then conclude that

$$|\partial_i u(x)| \leq \frac{n}{\alpha(n)\,r^{n+1}}\,\|u\|_{L^1(B(x,2r))}$$

   provided $B(x,2r) \subset U$.

   Hint: Mean value formula. (10+5)

3. Let $U \subset \mathbb{R}^n$ be open and bounded. Consider the Poisson equation with so-called Neumann boundary conditions,

$$\begin{aligned} -\Delta u &= f \quad \text{in } U\,, \\ \nu \cdot Du &= 0 \quad \text{on } \partial U\,. \end{aligned}$$

Show that this equation cannot have a solution unless

$$\int_U f \, dx = 0\,.$$

(10)

4. A function $u \in L^1_{\text{loc}}(\mathbb{R} \times \mathbb{R})$ is called a *weak solution* of the wave equation on the line if

$$\int_{\mathbb{R}} \int_{\mathbb{R}} u(x,t) \, \big(v_{tt}(x,t) - v_{xx}(x,t) \big) \, dx \, dt = 0$$

for every $v \in C_0^\infty(\mathbb{R} \times \mathbb{R})$.

(a) Show that if $u \in C^2(\mathbb{R} \times \mathbb{R})$ is a classical solution of the wave equation, it is also a weak solution.

(b) Verify that

$$u(x,t) = \begin{cases} 1 & \text{for } x > t \\ 0 & \text{for } x \leq t \end{cases}$$

is a weak solution of the wave equation.

(5+10)

5. Consider the Korteweg–de Vries equation

$$u_t - 6uu_x + u_{xxx} = 0$$

on $\mathbb{T} \times (0, \infty)$. Show that

$$\begin{aligned} M &= \int_{\mathbb{T}} u \, dx\,, \\ E &= \int_{\mathbb{T}} u^2 \, dx\,, \end{aligned}$$

and

$$H = \int_{\mathbb{T}} \big(\tfrac{1}{2} \, u_x^2 + u^3 \big) \, dx$$

are all constants of the motion. (5+5+10)

6. Let H be a Hilbert space and u_n a sequence in H. Show that the following are equivalent.

   (i) $u_n \to u$ strongly;

   (ii) $u_n \rightharpoonup u$ weakly and $\|u_n\| \to \|u\|$.

(10+10)