Partial Differential Equations

Midterm Exam

Solutions

1. (a) Solve the partial differential equation

$$u_t + x^2\, u_x = 0\,,$$

where $u = u(x,t)$ on the open first quadrant of the (x,t) plane.
Hint: Show that $z(s) = u(x(s), s)$ is constant if $x'(s) = x^2(s)$.

(b) Draw the characteristic curves, then state a set of boundary and/or initial conditions that specify the solution uniquely in the first quadrant of the (x,t) plane.

(10+10)

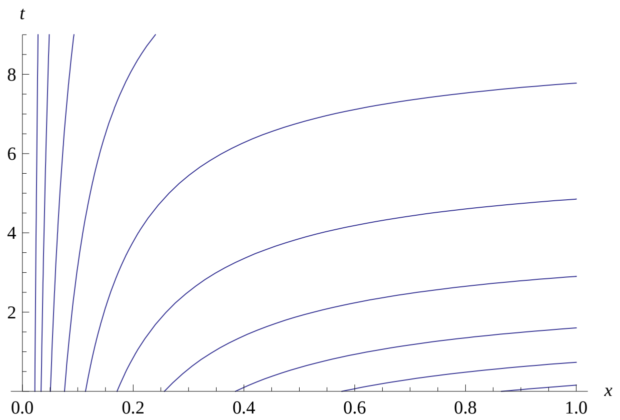
Solution:

$$z'(s) = u_x(x(s), s)\, x'(s) + u_t(x(s), s) = u_t + x^2\, u_x = 0\,.$$

Now the characteristic curves are the solutions of the ordinary differential equation $x'(s) = x^2(s)$:

$$\mathrm{d}t = \frac{\mathrm{d}x}{x^2}, \qquad s = x_0^{-1} - x^{-1}, \qquad x_0 = \frac{x}{1 + xs}.$$

The characteristic curve which passes through any point $(x,t) \equiv (x(s), s)$ in the (open) first quadrant also passes through the positive x-axis at $(x_0, 0) \equiv (x(0), 0)$:



Moreover, the t-axis is also a characteristic line. Thus, in the first quadrant, we need to specify initial data

$$u = g \qquad \text{on } [0,\infty) \times \{t = 0\}$$

and the solution is given by

$$u(x,t) = g\left(\frac{x}{1+xt}\right).$$

2. Let $U \subset \mathbb{R}^n$ open and bounded and $u \in C^2(\bar{U})$. We say that u is *superharmonic* if

$$-\Delta u \geq 0 \quad \text{in } U\,.$$

   (a) Prove that for superharmonic u,

$$u(x) \geq \fint_{B(x,r)} u(y)\,dy$$

   for all $B(x,r) \subset U$.

   Note: Do the computation explicitly. Simply referring to the homework is not enough.

   (b) Show that if u is superharmonic and $u \geq 0$ on ∂U, then $u \geq 0$ in $\bar{U}$.

(10+10)

Solution:

(a) Set

$$\phi(r) = \fint_{\partial B(x,r)} u(y)\,\mathrm{d}S(y) = \fint_{\partial B(x,1)} u(ry)\,\mathrm{d}S(y).$$

Then

$$\begin{aligned}
\phi'(r) &= \fint_{\partial B(x,1)} y \cdot Du(ry)\,\mathrm{d}S(y) \\
&= \fint_{\partial B(x,r)} \frac{z}{r} \cdot Du(z)\,\mathrm{d}S(z) \\
&= \frac{1}{n\alpha(n)r^{n-1}} \int_{B(x,r)} \Delta u(z)\,\mathrm{d}z \leq 0
\end{aligned}$$

The rest is to mention that $\lim\limits_{r\to 0} \phi(r) = u(x)$.

(b) Let us prove that $\min\limits_{\bar{U}} u = \min\limits_{\partial U} u$. Assume that $\min\limits_{\bar{U}} u < \min\limits_{\partial U} u$ (it can not be greater, since $\partial U \subset \bar{U}$). Then $\exists x_0 \in U^0$ such that $u(x_0) = \min\limits_{\bar{U}} u$. For $B(x_0, r) \subset U$ we have (see above)

$$u(x_0) \geq \fint_{B(x_0,r)} u \, \mathrm{d}y \geq \min_{B(x_0,r)} u \, .$$

Hence u is constant on $B(x_0, r)$. Assuming that U is path-connected, for any $x \in U$ there exists a continuous curve $\gamma\colon [0,1] \to U$ with $\gamma(0) = x_0$ and $\gamma(1) = x$. Set

$$s^* = \sup\{s \in [0,1]\colon u(\gamma(t)) = \min_{\bar{U}} u \text{ for } t \in [0,s]\}$$

By continuity, $u(\gamma(s^*)) = \min u$. Moreover, $s^* = 1$ because if not, we apply the first part of the argument with $x_0 = \gamma(s^*)$ and arrive at a contradiction to the maximality of s^*.

3. Recall that the solution to the heat equation

$$\begin{aligned} u_t - \Delta u = 0 \quad &\text{in } \mathbb{R} \times (0, \infty)\,, \\ u = g \quad &\text{on } \mathbb{R} \times \{t = 0\} \end{aligned}$$

is given by

$$u(x,t) = \int_{\mathbb{R}} \Phi(x - y, t) \, g(y) \, dy \, ,$$

where, for $t > 0$,

$$\Phi(z,t) = \frac{1}{\sqrt{4\pi t}} \, e^{-\frac{|z|^2}{4t}} \, .$$

(a) Show that if g is an even function, i.e. if $g(x) = g(-x)$, then $u(\cdot, t)$ is even for every $t \geq 0$.

(b) What does this imply for the solution of the heat equation on the halfline with Neumann boundary conditions, i.e.

$$\begin{aligned} u_t - \Delta u = 0 \quad &\text{in } (0, \infty) \times (0, \infty)\,, \\ u_x = 0 \quad &\text{on } \{x = 0\} \times (0, \infty)\,, \\ u = g \quad &\text{on } (0, \infty) \times \{t = 0\}\,? \end{aligned}$$

(10+10)

Solution:

(a)

$$\begin{aligned} u(x,t) &= \int_{\mathbb{R}} \Phi(x-y,t)\, g(y)\, dy \\ &= \int_{\mathbb{R}} \Phi(x-y,t)\, g(-y)\, dy \\ &= \int_{\mathbb{R}} \Phi(x+z,t)\, g(z)\, dz \\ &= \int_{\mathbb{R}} \Phi(-(-x-z),t)\, g(z)\, dz \\ &= \int_{\mathbb{R}} \Phi(-x-z,t)\, g(z)\, dz = u(-x,t). \end{aligned}$$

(b) The problem is equivalent to

$$\begin{aligned} u_t - \Delta u = 0 \quad &\text{in } \mathbb{R} \times (0,\infty)\,, \\ u(x,0) = g(x) \quad &\text{on } (0,\infty) \times \{t=0\}\,, \\ u(x,0) = g(-x) \quad &\text{on } (-\infty,0) \times \{t=0\}\,. \end{aligned}$$

Theorem 1 [Evans, p.47] shows that $u \in C^\infty\ \forall\, t > 0$. Hence $u_x(0,t)$ is well defined $\forall\, t > 0$ and is zero because u is even.

4. Let $U \subset \mathbb{R}^n$ open and bounded. Recall that $1 \le p < \infty$, the L^p-norm of a suitably integrable function u is defined

$$\|u\|_{L^p}^p = \int_U |u(x)|^p\, dx$$

while the L^∞-norm is given by

$$\|u\|_{L^\infty} = \operatorname*{ess\,sup}_{x \in U} |u(x)|\,.$$

(a) Show that if $u \in C_1^2(\bar{U} \times [0,\infty))$ satisfies the heat equation

$$\begin{aligned} u_t - \Delta u = 0 \quad &\text{in } U \times (0,\infty)\,, \\ u = g \quad &\text{on } U \times \{t=0\}\,, \\ \nu \cdot Du = 0 \quad &\text{on } \partial U \times (0,\infty)\,, \end{aligned}$$

and $p \ge 2$, then

$$\|u(\cdot,t)\|_{L^p} \le \|g\|_{L^p}\,.$$

Note: For simplicity, you may assume that p is an even integer.

(b) Prove that, provided $u \in C(\bar{U})$,

$$\lim_{p\to\infty} \|u\|_{L^p} = \|u\|_{L^\infty} .$$

Note: The statement is also true if u is only L^∞, but the proof is not so elementary.

(c) Explain why (a) and (b) imply yet another proof of the maximum principle for the heat equation.

(10+10+10)

Solution:

(a)

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u(x,t)\|^p_{L^p} = \frac{\mathrm{d}}{\mathrm{d}t}\int_U |u(x)|^p\,dx = p\int_U |u(x,t)|^{p-1}u_t(x,t)\,dx \; =$$
$$= p\int_U |u(x,t)|^{p-1}\Delta u(x,t)\,dx = -p(p-1)\int_U |u(x,t)|^{p-2}|Du(x,t)|^2\,dx \; +$$
$$+ \int_{\partial U} |u(x,t)|^{p-2}\nu\cdot Du(x,t)\,dx = -p(p-1)\int_U |u(x,t)|^{p-2}|Du(x,t)|^2\,dx \le 0.$$
$$\|u(x,0)\|^p_{L^p} = \|g(x)\|^p_{L^p}.$$

(b)

$$\|u\|_{L^p} \le \left(\int_U \|u\|^p_{L^\infty}\,dx\right)^{1/p} = \|u\|_{L^\infty}\cdot\mu(U)^{1/p} \to \|u\|_{L^\infty} \qquad \text{as } p\to\infty.$$

Then $\forall\; 0 < M < \|u\|_{L^\infty}$ consider $U_M := \{|u| > M\}$. The set is nonempty, since $u \in C(\bar{U})$. Then

$$\|u\|_{L^p} \ge \left(\int_{U_M} |u(x)|^p\,dx\right)^{1/p} \ge \left(\int_{U_M} |M|^p\,dx\right)^{1/p} = M\cdot\mu(U_M)^{1/p} \to M \qquad \text{as } p\to\infty.$$

Since M can be chosen arbitrarily close to $\|u\|_{L^\infty}$, the statement follows.

(c) Let $p\to\infty$.

5. Show that for solutions of the wave equation

$$u_{tt} - \Delta u = 0$$

on $\mathbb{R}^n \times (0,\infty)$, where $u(\,\cdot\,,0)$ has compact support, the energy

$$E(t) = \int_{\mathbb{R}^n} \left(u_t^2 + |Du|^2\right)dx$$

is constant in time.

(10)

Solution:

$$\begin{aligned}\frac{1}{2}\dot{E}(t) &= \int_{\mathbb{R}^n} \big(u_t u_{tt} + Du \cdot Du_t\big)\,\mathrm{d}x \\ &\lim_{R\to\infty} \int\limits_{\partial B(0,R)} \nu \cdot Du \cdot u_t \,\mathrm{d}S + \int_{\mathbb{R}^n} \big(u_t \Delta u - \Delta u u_t\big)\, dx \\ &= 0.\end{aligned}$$

From the first tot he second line, we used integration by parts. The first term in the second line vanishes due to the assumption on the compact support of the initial data which, due to the finite speed of propagation of the wave equation, remains compactly supported for all times $t > 0$, the second term vanishes when plugging in the wave equation.