General Mathematics and Computational Science I

Exercise 3

September 11, 2007

1. (Repeat from Exercise Sheet 2.)

   Let X be a nonempty set with an *equivalence relation* $\sim$ on it.

   Recall that this means that

   (i) $x \sim x$ for every $x \in X$ ("reflexivity"),

   (ii) $x \sim y$ implies that $y \sim x$ for $x, y \in X$ ("symmetry")

   (iii) $x \sim y$ and $y \sim z$ implies $x \sim z$ for $x, y, z \in X$ ("transitivity").

   Also recall that for any $x \in X$, we define the *equivalence class* represented by x as

   $$[x] = \{y \in X : y \sim x\} .$$

   Prove that for all $x, y \in X$,

   $$[x] = [y] \qquad \text{if and only if} \qquad x \sim y .$$

2. Check whether each of the following relations is an equivalence relation, i.e. check whether it is reflexive, symmetric, and transitive. If a property holds, prove that it does. If a property does not hold, give a counter example.

   (a) On $\mathbb{Z}$, define $x \sim y$ if and only if $x - y$ is divisible by 3.
   Note: Any number $n \in \mathbb{Z}$ is divisible by 3 if there exists $k \in \mathbb{Z}$ such that $n = 3 \cdot k$.

   (b) Let X be a nonempty set. Define, for any two subsets $A, B \subseteq X$, that $A \sim B$ if and only if $A \subseteq B$.

   (c) On $\mathbb{Z} \times \mathbb{Z}$, define $(a, b) \sim (a', b')$ if and only if $ab' = ba'$.

3. Recall from class that we studied a binary operation $F : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ with the following properties:

   (A1) $F(a, 1) = s(a)$ for all $a \in \mathbb{N}$,

   (A2) $F(a, s(b)) = s(F(a, b))$ for all $a, b \in \mathbb{N}$, where $s\colon \mathbb{N} \to \mathbb{N}$ is as in Peano's axioms.

Prove that if

$$F(a, c) = F(b, c)$$

for some $a, b, c \in \mathbb{N}$, then $a = b$.