General Mathematics and Computational Science I

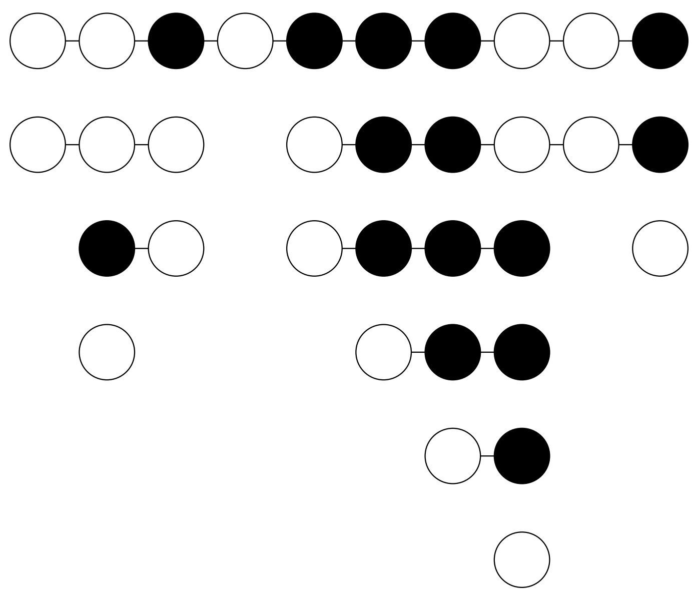
Midterm I

October 4, 2007

1. Prove, by induction, that $2^{2n} - 1$ is divisible by 3 for every $n \in \mathbb{N}$. (8)

2. Consider the game where n balls are arranged along a chain; k balls are white and the remaining $n - k$ balls are black. You cannot remove a black ball. When removing a white ball, the chain breaks into two pieces and the former neighbors of the removed ball change their color.

   It is sometimes, but not always, possible to remove all balls following these rules. The following example shows a case where it can be done.



   (a) Conjecture a condition which ensures that all balls can be removed in accordance with the rules above.

   (b) Prove, by induction or otherwise, that your condition is sufficient.

   (c) Prove, by induction or otherwise, that your condition is necessary.

(3+3+3)

3. For $a, b \in \mathbb{Z}$, define $a \sim b$ if and only if $|a|^2 = |b|^2$.

   Check whether this relation is an equivalence relation, i.e. check whether it is reflexive, symmetric, and transitive. If a property holds, prove that it does. If a property does not hold, give a counter example. (8)

4. Let $(\mathbb{N}, s, 1)$ be a set with successor map s and distinguished element 1 satisfying the Peano axioms, and let $(\mathbb{N}', s', 1')$ be a different set with its own successor map s' and distinguished element $1'$ also satisfying the Peano axioms.

   Assume that $\Phi \colon \mathbb{N} \to \mathbb{N}'$ is a map with $\Phi(1) = 1'$ and $\Phi(s(n)) = s'(\Phi(n))$ for every $n \in \mathbb{N}$. Prove that Φ is surjective. (8)

5. In how many different ways can 5 boys and 5 girls sit in (a) a row and (b) a circle so that no boy sits next to a boy and no girl sits next to a girl? (6+3)

6. Show that
$$\binom{n}{k}\binom{n-k}{j} = \binom{n}{j}\binom{n-j}{k}.$$
(8)