Applied Analysis

Homework 1

due September 14, 2007

1. Let $D \subset \mathbb{C}$, $z_0 \in \overline{D}$, and $f, g \colon D \to \mathbb{C}$. Show that
$$f(z) = O(g(z)) \quad \text{as } z \to z_0 \text{ from } D\,,$$
according to the definition given in class, if and only if
$$\limsup_{z \to z_0} \left| \frac{f(z)}{g(z)} \right| < \infty\,.$$

2. (Cf. Miller, p. 18.) Suppose $p > 0$ and $z > 0$. Show that
$$\ln z = o(z^p) \quad \text{as } z \to \infty\,.$$

3. (Cf. Miller, p. 19.) Suppose $q > 0$ and $z > 0$. Show that
$$z^q = O(\mathrm{e}^z)$$

4. (Cf. Miller, p. 20.) Give an example to show that in general
$$a(z)\,(1 + O(g(z))) + b(z)\,(1 + O(g(z))) \neq (a(z) + b(z))\,(1 + O(g(z)))\,.$$

5. Show, by example, that the statement
$$f(z) = O(g(z)) \quad \text{as } z \to z_0 \text{ from } D$$
contains absolutely no information about the magnitude of $f'(z)$.

6. Recall the asymptotic series for the exponential integral
$$\mathrm{Ei}(z) = \int_z^\infty \frac{\mathrm{e}^{-t}}{t}\,\mathrm{d}t\,,$$
where
$$\mathrm{Ei}(z) = \mathrm{e}^{-z} \left[\frac{1}{z} - \frac{1}{z^2} + \frac{2!}{z^3} - \frac{3!}{z^4} + \cdots + \frac{(-1)^{n+1}\,(n-1)!}{z^n} \right] + R_n(z)$$
with
$$R_n(z) = (-1)^n\,n! \int_z^\infty \mathrm{e}^{-1}\,t^{-n-1}\,\mathrm{d}t\,.$$

(a) Find a (near-)optimal truncation $n(z)$ as a function of z.

Note: It is sufficient to derive the leading order behavior of $n(z)$. You will find Stirling's approximation useful.

(b) Give an estimate for $|R_{n(z)}(z)|$ and describe its behavior.